Math32a/2, R. Kozhan

Midterm2

Nov 19, 2012

NAME: Solutions

UID: ______________________________

Circle your TA and discussion session:

2A-Tues-Jordan Greenblatt 2C-Tues-William Rosenbaum 2E-Tues-Silas Richelson

2B-Thur-Jordan Greenblatt 2D-Thur-William Rosenbaum 2F-Tues-Silas Richelson

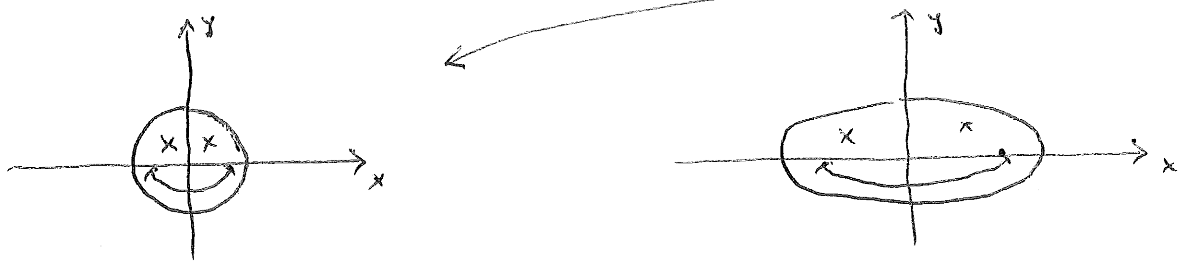
Instructions:

- If you get stuck, move on to the next question! You don't have a lot of time.
- Show all work if you want to get full credit.
- You have an extra scratch paper on the last page. If you need more raise your hand and ask.
- No books, notes, electronics (incl. calculators and cell-phones) are allowed.
- Good luck and have fun!

Question	Max	Your score
1	11	
2	12	
3	8	
4	12	
Total	43	

Problem 1

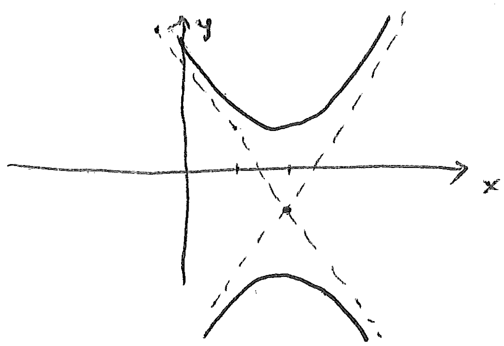
(a) (2 points) Suppose the set of all points (x, y) that satisfy the equation $F(x, y) = 0$ is as on the picture. Sketch the set of all points (x, y) that satisfy the equation $F(x/2, y) = 0$.



(b) (3 points) Find the center, asymptotes, and sketch the hyperbola $(x-2)^2 - \frac{(y+1)^2}{4} = -4$

Center : $(2, -1)$

Asymptotes : $(x-2)^2 = \frac{(y+1)^2}{4} \Rightarrow y = -1 \pm 2(x-2)$



(c) (2 points) Complete the definition: $z = f(x, y)$ is called differentiable at (a, b) if...

$\Delta z = f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y + e(\Delta x, \Delta y)$, where

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{e(\Delta x, \Delta y)}{\sqrt{\Delta x^2 + \Delta y^2}} = 0$$

(d) (4 points) Is the following statement necessarily TRUE of could it be FALSE? No justification is needed.

(i) If $f(x, y) \to L$ as $(x, y) \to (a, b)$ along every straight line through (a, b), then

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

FALSE

(limit along every line is not enough. you need every curve too)

(ii) If both $f_x(a, b)$ and $f_y(a, b)$ exist then f is differentiable at (a, b) $\longrightarrow$ FALSE

(not true: theorem needs f_x, f_y to exist in a disk around (a,b), and be continuous at (a,b))

(iii) $\lim_{y\to b} \frac{f(a,y)-f(a,b)}{y-b} = f_y(a, b)$ $\longrightarrow$ TRUE

if you take $y = b+h$ then you obtain the usual definition of f_y

(iv) If $\vec{r}(t)$ is the law of motion of a particle in space, then $\vec{r}\,'(t)$ is its speed, and $||\vec{r}\,'(t)||$ is its velocity. FALSE

Problem 2

Compute the following limits or prove that they do not exist. Justify all the steps.

(a) (4 points)

$$\lim_{(x,y)\to(0,0)} \frac{|x+y|}{|x|+|y|}$$

Check two paths going through $(0,0)$:

$x=t$, $y=0$: $\lim_{t\to 0} \frac{|t|}{|t|} = 1$

$x=t$, $y=-t$: $\lim_{t\to 0} \frac{0}{2|t|} = 0$

Since $0 \neq 1$, two different paths give two different limits so the limit does not exist in $\mathbb{R}^2$

(b) (4 points)

$$\lim_{(x,y)\to(0,0)} \frac{e^{x+y}}{x-y-1}$$

$x+y$ is a polynomial and thus continuous e^t is continuous so e^{x+y} is continuous because compositions of continuous functions are continuous. $x-y-1$ is also a polynomial and thus continuous and the quotient of two continuous functions $\frac{f(x,y)}{g(x,y)}$ is continuous at (a,b) as long as $g(a,b) \neq 0$. Since $0-0-1=-1 \neq 0$ at $(0,0)$, $\frac{e^{x+y}}{x-y-1}$ is continuous at $(0,0)$

so $\lim_{(x,y)\to(0,0)} \frac{e^{x+y}}{x-y-1} = \frac{e^{0+0}}{0-0-1} = \frac{1}{-1} = -1$

(c) (4 points)

$$\lim_{(x,y)\to(0,0)} x \sin\left(x^{2012} - \frac{1}{y^{2013}}\right)$$

The sine function is bounded between -1 and 1 so $-1 \leq \sin\left(x^{2012} - \frac{1}{y^{2013}}\right) \leq 1$ for any $(x,y) \neq (0,0)$. Then

$-|x| \leq x\sin\left(x^{2012} - \frac{1}{y^{2013}}\right) \leq |x|$ so by the squeeze theorem (since $\lim_{(x,y)\to(0,0)} -|x| = 0$ and $\lim_{(x,y)\to(0,0)} |x| = 0$), $\lim_{(x,y)\to(0,0)} x\sin\left(x^{2012} - \frac{1}{y^{2013}}\right)$

$= 0$.

Problem 3

(8 points) Use linear approximation to approximate $\sqrt[3]{8.04}\sqrt{8.94}$ (Hint: $\sqrt[3]{x}\sqrt{y}$).

Let $f(x,y) = \sqrt[3]{x} \cdot \sqrt{y}$

Let $a = 8$, $\Delta x = 0.04$

$b = 9$, $\Delta y = -0.06$

We need to approximate $f(a+\Delta x, b+\Delta y)$.

Use linearization:

$$L(a+\Delta x, b+\Delta y) = f(a,b) + f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y$$

$$f(a,b) = \sqrt[3]{8}\sqrt{9} = 2 \cdot 3 = 6$$

$$f_x(x,y) = \frac{1}{3} x^{-2/3} \sqrt{y} \Rightarrow f_x(a,b) = \frac{1}{3} \frac{1}{\sqrt[3]{8^2}} \cdot \sqrt{9} = \frac{1}{4}$$

$$f_y(x,y) = \sqrt[3]{x} \frac{1}{2\sqrt{y}} \Rightarrow f_y(a,b) = 2 \cdot \frac{1}{2 \cdot 3} = \frac{1}{3}$$

So

$$\sqrt[3]{8.04}\sqrt{8.94} = f(8.04, 8.94) = f(a+\Delta x, b+\Delta y) \approx$$

$$\approx L(a+\Delta x, b+\Delta y) = 6 + \frac{1}{4} \cdot 0.04 - \frac{1}{3} \cdot 0.06 = 5.99$$

Problem 4

Consider the equation $z^2 = y - x^2 + 2x$.

(a) (3 points) Classify and sketch the surface.

(b) (3 points) Find its center and the parametric equation of its axis of symmetry.

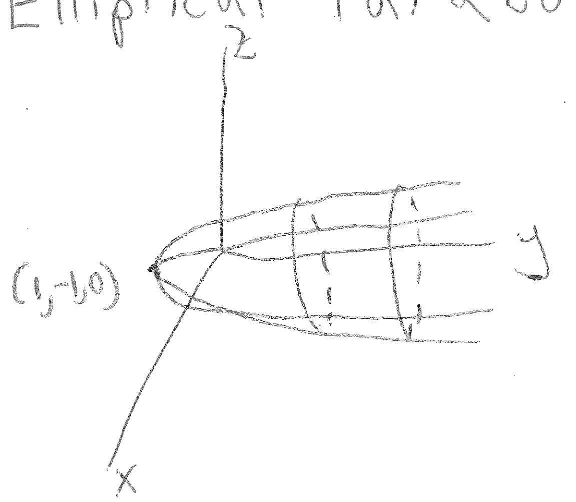
(c) (2 points) Classify all horizontal traces of this surface (i.e. traces parallel to the xy coordinate plane).

(d) (4 points) Consider the horizontal trace of this surface that passes through the point $(0, 1, -1)$. Find the **parametric** equation of the tangent **line** to this **trace** at the point $(0, 1, -1)$.

(a) Elliptical Paraboloid: $z^2 = y - x^2 + 2x$

$\Rightarrow y = (x^2 - 2x) + z^2$

$\Rightarrow y = (x-1)^2 + z^2 - 1$



(b) Center: $(1, -1, 0)$

Axis of Symmetry: $x(t) = 1$, $y(t) = t - 1$, $z(t) = 0$

(c) If $z = C$ (a constant) $y = (x-1)^2 + (C^2 - 1)$ so these are parabolas.

(d) $z = -1$ is the plane containing the tangent line. Its direction vector (in $\mathbb{R}^3$) is $\langle 1, \frac{dy}{dx}\big|_{x=0}, 0\rangle$ (when we view y as a function of x in the plane $z = -1$). In this plane: $y = (x-1)^2$ so $\frac{dy}{dx} = 2x - 2 \Rightarrow \frac{dy}{dx}\big|_{x=0} = -2$ so this vector $\langle 1, -2, 0\rangle$. Parameterizing yields: $x(t) = t$, $y(t) = -2t + 1$, $z(t) = -1$

ANOTHER SOL.:
WHEN $z = -1$ WE GET
$y - x^2 + 2x = 1$
$y = x^2 - 2x + 1$
Its tangent line at $x = 0$ is
$y - 1 = -2x$
Parametric form: $\begin{cases} x = t \\ y = 1 - 2t \end{cases}$
in $\mathbb{R}^2$. But we're in $\mathbb{R}^3$, so: $\begin{cases} x = t \\ y = 1 - 2t \end{cases}$