Math32a/1, R. Kozhan
Midterm 1
Oct 24, 2011

Name:________________________________
UID:________________________________
Section/TA:________________________

Instructions:

- If you get stuck, move on to the next question! You don't have a lot of time.
- Make sure to look at the last question. It's short and easy. Free points!
- Show all work if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer even if your final answer is correct.
- No books, notes, electronics (incl. calculators and cell-phones) are allowed.
- Good luck!

Question	Max	Your score
1	12	
2	8	
3	14	
4	10	
5	10	
6	14	
Total	68	

Problem 1

(a) (4 points) Find the center and the radius of the circle $x^2 + y^2 - 4x + 10y = 0$.

(b) (8 points) Find parametric equation for the path of a particle that moves around this circle **twice counterclockwise, starting from the top point of the circle**.

Solution.
(a) Note that

$$x^2 + y^2 - 4x + 10y = (x^2 - 4x + 4) - 4 + (y^2 + 10y + 25) - 25 = (x-2)^2 + (y+5)^2 - 29$$

So we get equation $(x-2)^2 + (y+5)^2 = 29$ which is the equation of the circle of radius $\sqrt{29}$ and center $(2, -5)$.

(b) Note that

$$x = \cos t$$
$$y = \sin t, \qquad 0 \le t \le 2\pi$$

is a parametrization of the circle of radius 1 with center at the origin, which goes around once counterclockwise starting with the rightmost point. Note that we're in the topmost point at time $t = \pi/2$. Therefore

$$x = \cos t$$
$$y = \sin t, \qquad \frac{\pi}{2} \le t \le \frac{\pi}{2} + 2\pi$$

gives the same circle which starts at the top and travels once counterclockwise. To travel twice around we can just extend the time interval by another 2π:

$$x = \cos t$$
$$y = \sin t, \qquad \frac{\pi}{2} \le t \le \frac{\pi}{2} + 4\pi$$

Finally we need to scale this circle to have radius $\sqrt{29}$ and shift it to have center $(2, -5)$, so the parametrization becomes

$$x = \sqrt{29}\cos t + 2$$
$$y = \sqrt{29}\sin t - 5, \qquad \frac{\pi}{2} \le t \le \frac{\pi}{2} + 4\pi$$

One could also start with

$$x = \sin t$$
$$y = \cos t, \qquad 0 \le t \le 2\pi$$

but then notice that this parametrization is clockwise, so we'll need to put $-t$ instead of t to get clockwise orientation:

$$x = \sin(-t)$$
$$y = \cos(-t), \qquad 0 \le t \le 2\pi$$

and the end-result would be

$$x = -\sqrt{29}\sin t + 2$$
$$y = \sqrt{29}\cos t - 5, \qquad 0 \le t \le 4\pi$$

Problem 2

(8 points) Find the area of the triangle with vertices $P(1,2,-1)$, $Q(2,3,0)$, $R(-1,0,1)$.

Solution. Note that $\overrightarrow{PQ} = \langle 1,1,1\rangle$ and $\overrightarrow{PR} = \langle -2,-2,2\rangle$, so the area of the parallelogram built on those two vectors is

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = |4\vec{i} - 4\vec{j}| = \sqrt{4^2+4^2} = 4\sqrt{2}$$

So the area of the triangle is $\frac{1}{2}4\sqrt{2} = 2\sqrt{2}$.

Problem 3

(a) (4 points) Find parametric and symmetric equations of the line through the point $A(2,3,1)$ which is perpendicular to the plane $2y - z = 0$.

(b) (2 points) Prove that this line is parallel to the line

$$\begin{aligned} x &= 0 \\ y &= 2 + 4t \\ z &= -1 - 2t \end{aligned}$$

(c) (8 points) Find the equation of the unique plane passing through both of the above lines.

Solution.

(a) We are looking for the line through the point $(2,3,1)$ parallel to $\langle 0,2,-1\rangle$. Its parametric equation is

$$\begin{aligned} x &= 2 \\ y &= 3 + 2t \\ z &= 1 - t \end{aligned}$$

Its symmetric equation is

$$x = 2 \quad \text{and} \quad \frac{y-3}{2} = \frac{z-1}{-1}$$

(note that you have to keep $x = 2$)

(b) The line in (a) is parallel to $\langle 0,2,-1\rangle$. The line in (b) is parallel to $\langle 0,4,-2\rangle$. Since these two vectors are scalar multiples of each other, they are parallel. Therefore the lines are parallel.

(c) In order to get the normal vector of the plane we need to take the cross product of any two *non-parallel* vectors on the plane. One vector could be $\langle 0,2,-1\rangle$. The other can't be $\langle 0,4,-2\rangle$ since it's parallel to the previous one. Instead we can take the vector joining the two points on the two lines: $A(2,3,1)$ and $B(0,2,-1)$. So, $\overrightarrow{AB} = \langle -2,-1,-2\rangle$, and therefore a normal vector to the plane is

$$\langle 0,2,-1\rangle \times \langle -2,-1,-2\rangle = -5\vec{i} + 2\vec{j} + 4\vec{k}$$

So now we can just write the equation of the plane with normal vector $\langle -5,2,4\rangle$ through the point $A(2,3,1)$ (or, $B(0,2,-1)$ if you wish):

$$-5(x-2) + 2(y-3) + 4(z-1) = 0$$

i.e. $-5x + 2y + 4z = 0$.

Problem 4

(10 points) Find the unit tangent vector $\vec{T}$ and the principle normal vector $\vec{N}$ at time $t = \pi/2$ of the curve $\vec{r}(t) = \langle e^t \sin t, e^t \cos t, e^t \rangle$.

Solution.
$\vec{r}(t) = \langle e^t \sin t, e^t \cos t, e^t \rangle$, so

$$\vec{r}\,'(t) = \langle e^t \sin t + e^t \cos t, e^t \cos t - e^t \sin t, e^t \rangle$$

Its length:

$$\begin{aligned} |\vec{r}\,'(t)| &= \sqrt{(e^t \sin t + e^t \cos t)^2 + (e^t \cos t - e^t \sin t)^2 + e^{2t}} \\ &= \sqrt{2e^{2t} \sin^2 t + 2e^{2t} \cos^2 t + e^{2t}} = \sqrt{3} e^t \end{aligned}$$

So unit tangent vector is

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \left\langle \frac{\sin t + \cos t}{\sqrt{3}}, \frac{\cos t - \sin t}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$$

The derivative of $\vec{T}(t)$ is

$$\vec{T}\,'(t) = \left\langle \frac{\cos t - \sin t}{\sqrt{3}}, \frac{-\sin t - \cos t}{\sqrt{3}}, 0 \right\rangle$$

and its length is

$$|\vec{T}\,'(t)| = \frac{1}{\sqrt{3}} \sqrt{(\cos t - \sin t)^2 + (-\sin t - \cos t)^2} = \frac{1}{\sqrt{3}} \sqrt{2},$$

so the principle normal vector is

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|} = \left\langle \frac{\cos t - \sin t}{\sqrt{2}}, \frac{-\sin t - \cos t}{\sqrt{2}}, 0 \right\rangle$$

Finally, at time $t = \pi/2$:

$$\vec{T}(\pi/2) = \left\langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$$

$$\vec{N}(\pi/2) = \left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right\rangle$$

Problem 5

(10 points) Suppose $\vec{r}''(t) = 2\vec{j} - 6t\vec{k}$, $\vec{r}'(0) = 5\vec{k}$, $\vec{r}(0) = 2\vec{i}$. Find $\vec{r}(1)$.

Solution. Integrating $\vec{r}''(t) = 2\vec{j} - 6t\vec{k}$ we get:

$$\vec{r}'(t) = 2t\vec{j} - 3t^2\vec{k} + \vec{c}$$

for some **vector** $\vec{c} = \langle c_1, c_2, c_3 \rangle$. Plugging in $t = 0$, we obtain $\vec{c} = \vec{r}'(0)$. So $\vec{c} = 5\vec{k}$, and

$$\vec{r}'(t) = 2t\vec{j} + (5 - 3t^2)\vec{k}$$

Integrating this equation, we get

$$\vec{r}(t) = t^2\vec{j} + (5t - t^3)\vec{k} + \vec{w}$$

for some **vector** $\vec{w} = \langle w_1, w_2, w_3 \rangle$. Plugging in $t = 0$, we get $\vec{w} = \vec{r}(0) = 2\vec{i}$, so

$$\vec{r}(t) = 2\vec{i} + t^2\vec{j} + (5t - t^3)\vec{k}$$

Finally, at $t = 1$:

$$\vec{r}(1) = 2\vec{i} + \vec{j} + 4\vec{k}$$

Problem 6

(a) (2 points) Write the formula for the vector projection of $\vec{a}$ onto $\vec{b}$.

ANSWER:

$$\text{proj}_{\vec{b}}\,\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\,\frac{\vec{b}}{|\vec{b}|}$$

(b) (2 points) Let $x(t) = \cos t, y(t) = \sin(t)$. Find $\frac{dy}{dx}$ at $t = \pi/4$.

ANSWER:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t}{-\sin t},$$

so at $t = \pi/4$ we get $\frac{1/\sqrt{2}}{-1/\sqrt{2}} = -1$

Answer if the following statement is always TRUE or if it could be FALSE. No justification is needed.

(c) (2 points) Two lines in $\mathbb{R}^3$ perpendicular to a given line are parallel.

ANSWER: Could be FALSE: e.g. x-, y-, z- axes are perpendicular to each other but not parallel.

(d) (2 points) Two planes in $\mathbb{R}^3$ perpendicular to a given line are parallel.

ANSWER: TRUE.

(e) (2 points) Suppose $\vec{a}, \vec{b}, \vec{c}$ are nonzero vectors in $\mathbb{R}^3$. If $\vec{a}\times\vec{b} = \vec{a}\times\vec{c}$ then $\vec{b} = \vec{c}$.

ANSWER: Could be FALSE. E.g. $\vec{i}\times\vec{i} = \vec{i}\times 2\vec{i}$ (both are zeros) but $\vec{i} \neq 2\vec{i}$

(f) (2 points) Suppose $\vec{a}, \vec{b}, \vec{c}$ are nonzero vectors in $\mathbb{R}^3$. If $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$ then $\vec{b} = \vec{c}$.

ANSWER: Could be FALSE. E.g. $\vec{i}\cdot\vec{j} = \vec{i}\cdot 2\vec{j}$ (both are zeros) but $\vec{j} \neq 2\vec{j}$

(g) (2 points) Which one is true:

(i) If $\text{proj}_{\vec{b}}\,\vec{a} = \vec{a}$ then $\vec{a}\times\vec{b} = \vec{0}$.

(ii) If $\text{proj}_{\vec{a}}\,\vec{b} = \vec{a}$ then $\vec{a}\times\vec{b} = \vec{0}$.

ANSWER: $\text{proj}_{\vec{b}}\,\vec{a} = \vec{a}$ is equivalent to $\vec{a}$ and $\vec{b}$ being parallel. $\text{proj}_{\vec{a}}\,\vec{b} = \vec{a}$ just says that $\vec{b}$ equals to $\vec{a}$ plus some vector orthogonal to $\vec{a}$. Thus (i) is TRUE while (ii) could be FALSE.

Extra scratch paper