ID: 2c3491c8

1. Let $\mathbf{x}(t) = (\cosh(t), t)$ where $\cosh(t) = \frac{1}{2}(e^t + e^{-t})$.

(a) (/2) Find the velocity, $\dot{\mathbf{x}}(t)$.

$$x(t) = \left(\tfrac{1}{2}(e^t + e^{-t}), t\right)$$

$$x'(t) = \left(\tfrac{1}{2}(e^t - e^{-t}), 1\right)$$

(b) (/3) Find the speed.

$$\|x'(t)\| = \sqrt{\left(\tfrac{1}{2}e^t - \tfrac{1}{2}e^{-t}\right)^2 + 1}$$

$$= \sqrt{\tfrac{1}{4}(e^{2t} - 2 + e^{-2t}) + 1}$$

$$= \tfrac{1}{2}\sqrt{e^{2t} + e^{-2t} + 2}$$

$$= \boxed{\tfrac{1}{2}(e^t + e^{-t})}$$

(c) (/5) Compute the arclength, s, from $(1, 0)$ in direction $t > 0$. What is the x coordinate on the curve when $s = 1$?

$$S = \int_0^t \sqrt{\left(\tfrac{\partial x}{\partial t}\right)^2 + \left(\tfrac{\partial y}{\partial t}\right)^2}\, dt$$

$$S = \int_0^t \tfrac{1}{2}(e^t + e^{-t})\, dt$$

$$2s = (e^t - e^{-t})$$

$$\ln(2s) = \ln\left(e^t - \tfrac{1}{e^t}\right) = \ln\left((e^1 - e^{-1})^t\right)$$

$$\ln 2s = t\ln\left(e - \tfrac{1}{e}\right)$$

$$t = \frac{\ln\left(e - \frac{1}{e}\right)}{\ln(2s)} = \ln\left(e - \tfrac{1}{e} - 2\right)$$

I know this is wrong, but I have to isolate t somehow, then merci ♡

at $S = 1$, $t = \dfrac{\ln\left(e - \frac{1}{e}\right)}{\ln 2} = \ln\left(e - \tfrac{1}{e} - 2\right)$

$$X\left(\ln\left(e - \tfrac{1}{e} - 2\right)\right) = \left(\tfrac{1}{2}\left(e - \tfrac{1}{e} - 2 + \frac{1}{e - \frac{1}{e} - 2}\right), \ln\left(e - \tfrac{1}{e} - 2\right)\right)$$

ID: 203461089

2. (/10) Recall the Frenet-Serret formulae

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N},$$
$$\frac{d\mathbf{N}}{ds} = -\kappa\mathbf{T} + \tau\mathbf{B},$$
$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}.$$

Find $\frac{d^2\mathbf{T}}{ds^2}$, $\frac{d^2\mathbf{N}}{ds^2}$ and $\frac{d^2\mathbf{B}}{ds^2}$ in terms of the frame $\mathbf{T}, \mathbf{N}$ and $\mathbf{B}$ and the curvature, κ, torsion, τ and their derivatives, κ' and τ'.

$$\frac{d^2T}{ds^2} = \frac{d}{ds}\left(\kappa N\right)$$
$$= \kappa\frac{dN}{ds} + \kappa' N$$
$$= \kappa\left(-\kappa T + \tau B\right) + \kappa' N$$

$$\frac{d^2N}{ds^2} = -\left(\kappa' T + \kappa\frac{dT}{ds}\right) + \tau' B + \tau\frac{dB}{ds}$$
$$= -\kappa' T + \kappa^2 N + \tau' B - \tau^2 N$$

$$\frac{d^2B}{ds^2} = -\left(\tau' N + \tau\frac{dN}{ds}\right)$$
$$= -\tau' N + \tau\kappa T - \tau^2 B$$

ID: 2031477097

3. Let $f(x, y) = x^2y + xy^2$.

(a) (/5) From the definition find the directional derivative of f at the point $(1, 1)$ in the direction $(\frac{1}{2}, \frac{\sqrt{3}}{2})$.

$D_{\vec{u}} f(x,y) = f(u_x(x), u_y(y))$

$r(t) = t\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$f\left(\frac{1}{2}t, \frac{\sqrt{3}}{2}t\right) = \frac{t^2}{4} \cdot \frac{\sqrt{3}}{2}t + \frac{3t^2}{4} \cdot \frac{t}{2}$

$= \frac{t^3\sqrt{3}}{8} + \frac{3t^3}{8}$

$f(1,1) = 2, t = 2$

$= \frac{3\sqrt{3}+3}{2}$

(b) (/5) Repeat the above problem only this time use the fact that f is differentiable and the gradient of f to simplify the calculation. Be wary that two identical calculations wont get you 10 points.

$\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1$

it's a unit vector.

$\nabla f(x,y) = (2xy + y^2, x^2 + 2xy)$

$\nabla f(1,1) = (3, 3)$

$\nabla f(1,1) \cdot \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \frac{3}{2} + \frac{3\sqrt{3}}{2}$

$= \boxed{\frac{3+3\sqrt{3}}{2}}$

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4. (/10) Consider the quadric surface in $\mathbb{R}^3$ defined by the equation

$$-\frac{x^2}{4} - \frac{y^2}{9} + z^2 = 9.$$

Find the equation of the tangent plane of the quadric at the point, (8, 36, 13) by thinking of the quadric as the level set of an appropriately defined function.

$$F(x,y,z) = \frac{-x^2}{4} - \frac{y^2}{9} + z^2$$

$$\nabla F(x,y,z) = \left(\frac{-x}{2}, \frac{-2y}{9}, 2z\right) \quad \text{for level } c=9$$

$$\nabla F(8,36,13) = (-4,-8,26) \text{ , normal for plane}$$

$$\underset{\sim}{n} = (-2,-4,13)$$

plane:

$$-2x - 4y + 13z = (-2,-4,13)\cdot(8,36,13)$$

$$= -16 - 144 + 169$$

$$\boxed{-2x - 4y + 13z = 9}$$

$\overset{2}{36}$
$\times 4$
144

ID: 20034110094

5. (a) (/4) Consider a path parameterized by arclength, $\mathbf{x} = \mathbf{x}(s)$. Show that in this special case the curvature, κ, is the magnitude of the acceleration. That is:

$$\kappa = \|\mathbf{x}''(s)\|.$$

$\kappa = \left\| \frac{dT}{ds} \right\|$ by definition

$T = \frac{\frac{dx}{ds}}{\left\|\frac{dx}{ds}\right\|} =$

thus $T = \frac{dx}{ds}$

$\kappa = \left\| \frac{d}{ds}\left(\frac{dx}{ds}\right)\right\| = \left\|\frac{d^2x}{ds^2}\right\|$

(b) (/6) For a helix parametrized by

$$\mathbf{x}(t) = (\cos(t), \sin(t), t)$$

compute the torsion,

$$\tau = \frac{(\dot{\mathbf{x}} \times \ddot{\mathbf{x}}) \circ \dddot{\mathbf{x}}}{\|\dot{\mathbf{x}} \times \ddot{\mathbf{x}}\|^2}.$$

$x'(t) = (-\sin t, \cos t, 1)$

$x''(t) = (-\cos t, -\sin t, 0)$

$x'''(t) = (\sin t, -\cos t, 0)$

$(x' \times x'') = (0 + \cos t, -(0 + \sin t), \sin^2 t + \cos^2 t)$

$= (\cos t, -\sin t, 1)$

$\|x' \times x''\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

$(x' \times x'') \cdot x''' = \sin t \cos t + \sin t \cos t$

$\boxed{\tau = \sin t \cos t}$

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6. An object is said to move under the influence of a central force if its acceleration vector satisfies

$$\ddot{\mathbf{x}} = f(r)\mathbf{x}$$

where $r = \|\mathbf{x}\|$ and f is a real value differentiable function of one variable and $\mathbf{x} = \mathbf{x}(t)$ is the path of the object in $\mathbf{R}^3$.

(a) (/6) Show that the path of an object moving under the influence of a central force is in a plane that contains the origin. Hint: The key trick is to show that $\dot{\mathbf{x}} \times \mathbf{x}$ is a constant vector (i.e. does not change with time).

$$\frac{d}{dt}(\underset{\sim}{x}' \times \underset{\sim}{x}) = \underset{\sim}{x}'' \times \underset{\sim}{x} + \cancel{\underset{\sim}{x}' \times \underset{\sim}{x}'}^{\,0}$$

$$= \underset{\sim}{x}'' \times \underset{\sim}{x}$$

$$= f(r)\,\cancel{\underset{\sim}{x} \times \underset{\sim}{x}}^{\,0},$$

$$= 0, \text{ thus } \underset{\sim}{x}' \times \underset{\sim}{x} \text{ is constant.}$$

Since $\underset{\sim}{x}' \times x$ defines the motion perpendicular to velocity & position, it is a normal to a plane that encompasses the motion. If that normal is constant, the plane is constant.

If the motion begin at the origin, then that constant plane must contain that point.

P Q R

(b) (/4) If ~~an~~ the path of an object goes through the points, (1,2,3), (0,4,5) and (0,0,6), could the object be moving under the influence of a central force? Justify your answer with a calculation.

If the points are all in a plane intersecting the origin, it could be.

$\vec{PR} = (-1, -2, 3)$

$\vec{PQ} = (-1, 2, 2)$

$\vec{PR} \times \vec{PQ} = (-2+3, 6+4, -2-2)$

$= (1, 10, -4)$

$x + 10y - 4z = -24$, this object is not under the influence of a central force.

$0 + 0 - 0 \neq -24$