Midterm 2 Solutions

1

(a)

We want to find the length of the parametrized curve $\mathbf{r}$ from the initial point corresponding to time $t = 1$ to the point corresponding to a general time t, where t is in the interval $[1, \infty)$. We will denote this length by $s(t)$. To calculate $s(t)$, we use the formula

$$s(t) = \int_1^t ||\mathbf{r}'(\tau)||d\tau.$$

Thus we need to find $||\mathbf{r}'(\tau)||$. Since $\mathbf{r}'(\tau) = \langle 2\tau^{-1/2}, \tau^{-1}, 2\rangle$, we get

$$\begin{aligned} s||\mathbf{r}'(\tau)|| &= \sqrt{\frac{4}{\tau} + \frac{1}{\tau^2} + 4} \\ &= \sqrt{\frac{4\tau^2 + 4\tau + 1}{\tau^2}} \\ &= \sqrt{\frac{(2\tau + 1)^2}{\tau^2}} \\ &= \frac{2\tau + 1}{\tau} \\ &= 2 + \frac{1}{\tau}. \end{aligned}$$

Therefore,

$$\begin{aligned} s(t) &= \int_1^t ||\mathbf{r}'(\tau)||d\tau \\ &= \int_1^t \left(2 + \frac{1}{\tau}\right) d\tau \\ &= 2t - 2 + \ln t. \end{aligned}$$

Notes: (1) Many students tried to calculate the integral

$$\int_1^\infty ||\mathbf{r}'(\tau)||d\tau,$$

rather than

$$\int_1^t ||\mathbf{r}'(\tau)||d\tau.$$

The former integral gives the length of the whole curve starting from time $t = 1$, which is infinite, as we can see from the arclength function $s(t) = 2t - 2 + \ln t$ ($s(t) \to \infty$ as $t \to \infty$). The former integral represents a single value, not a function. When you are asked to find the arclength *function*, your answer must be a *function*, with a variable input; the function gives the length of the curve from the initial point corresponding to $t = 1$ to the terminal point corresponding to the input time.

(2) Some students used the lower limit 0 in the integral rather than 1. Remember, the lower limit is the initial time from which you are measuring arclength, which in this case is $t = 1$. Just because it is the *initial* time, does not mean it is *zero*.

(3) Some students wrote something like the following:

$$\begin{aligned} s(t) &= \int_1^t ||\mathbf{r}'(\tau)||d\tau \\ &= \int_1^t ||\mathbf{r}'(1)||d\tau \\ &= \int_1^t 3d\tau \\ &= 3(t-1), \end{aligned}$$

which is incorrect. You do not evaluate an integral by evaluating the integrand (in this case $||\mathbf{r}'(\tau)||$) at an endpoint; you evaluate an integral by evaluating the *antiderivative* of the integrand at the endpoints. This may have just been a careless mistake, but since several students did it I wanted to point it out.

(b)

To find the arclength parametrization, we must express the time t as a function of the arclength s. We know that

$$s = g(t) = \frac{1}{6}((1 + 4t)^{3/2} - 1).$$

Solving for t, we get

$$\begin{aligned} s &= \frac{1}{6}((1+4t)^{3/2}-1) \\ \iff 6s &= (1+4t)^{3/2}-1 \\ \iff 6s+1 &= (1+4t)^{3/2} \\ \iff (6s+1)^{2/3} &= 1+4t \\ \iff t &= \frac{((6s+1)^{2/3}-1)}{4} = g^{-1}(s). \end{aligned}$$

Now, we insert this equation for t into the original parametrization $\mathbf{r}$, to get the arclength parametrization $\mathbf{r_1}$:

$$\begin{aligned} \mathbf{r_1}(s) &= \mathbf{r}(g^{-1}(s)) \\ &= \left\langle \frac{(6s+1)^{2/3}-1}{4}, \frac{2}{3}\left(\frac{((6s+1)^{2/3}-1)}{4}\right)^{3/2}, \frac{2}{\sqrt{3}}\left(\frac{((6s+1)^{2/3}-1)}{4}\right)^{3/2}\right\rangle. \end{aligned}$$

2

Using the curvature formula

$$\kappa(x) = \frac{|f''(x)|}{(1+f'(x)^2)^{3/2}},$$

we substitute $f(x) = e^{\alpha x}$ to get

$$\kappa(x) = \frac{\alpha^2 e^{\alpha x}}{(1+\alpha^2 e^{2\alpha x})^{3/2}}, \tag{1}$$

and then setting $x = 0$ we get

$$\kappa(0) = \frac{\alpha^2}{(1+\alpha^2)^{3/2}} \tag{2}$$

for the curvature of the graph of $y = e^{\alpha x}$ at $x = 0$. We want to find the value or values of α which maximize this curvature, so we differentiate with respect to α:

$$\begin{aligned} \frac{d}{d\alpha}\left(\frac{\alpha^2}{(1+\alpha^2)^{3/2}}\right) &= \frac{2\alpha(1+\alpha^2)^{3/2} - 3\alpha^3(1+\alpha^2)^{1/2}}{(1+\alpha^2)^3} \\ &= \frac{\alpha(2-\alpha^2)}{(1+\alpha^2)^{5/2}}. \end{aligned}$$

Setting this derivative equal to zero to find the critical points, we get $\alpha = 0$ and $\alpha = \pm\sqrt{2}$. Now for $\alpha = 0$, the curvature at the origin is zero, as we see by plugging $\alpha = 0$ into the formula (2) (which makes sense, since in that case the graph is a straight line). Since the curvature (2) is positive for all other values of α, this is not a maximum; so we can discard $\alpha = 0$. Since the curvature (2) goes to zero as α goes to positive infinity and negative infinity, the maximum curvature must occur at $\alpha = \sqrt{2}$ or $\alpha = -\sqrt{2}$. Substituting these values into the curvature formula (2) gives the same curvature, $2/(3^{3/2})$. So the maximum curvature occurs at both $\alpha = \sqrt{2}$ and $\alpha = -\sqrt{2}$.

Note: Some students tried to differentiate the function (1) above with respect to x, rather than differentiating the function (2) with respect to α. Remember, (1) gives the curvature of the graph of $y = e^{\alpha x}$ at a general point x, but we only care about the point $x = 0$. We would only differentiate (1) if we wanted to maximize the curvature of a *single* graph (corresponding to a single value of α) over *all* points x. Instead, we are trying to maximize the curvature at a *single* point ($x = 0$) over *all* graphs (corresponding to *all* values of α).

3

First we find the velocity and acceleration. We have

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 1, -\sin t, \cos t\rangle,$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = \langle 0, -\cos t, -\sin t\rangle.$$

Next, we use the formula

$$a_{\mathbf{T}}(t) = \mathbf{a}(t) \cdot \mathbf{T}(t) = \frac{\mathbf{a}(t) \cdot \mathbf{v}(t)}{||\mathbf{v}(t)||}$$

to find the tangential component of acceleration $a_{\mathbf{T}}(t)$:

$$\begin{aligned} a_{\mathbf{T}}(t) &= \frac{\mathbf{a}(t) \cdot \mathbf{v}(t)}{||\mathbf{v}(t)||} \\ &= \frac{\langle 0, -\cos t, -\sin t\rangle \cdot \langle 1, -\sin t, \cos t\rangle}{||\mathbf{v}(t)||} \\ &= \frac{0 + \sin t \cos t - \sin t \cos t}{||\mathbf{v}(t)||} \\ &= \frac{0}{||\mathbf{v}(t)||} \\ &= 0. \end{aligned}$$

So, $a_{\mathbf{T}}(t) = 0$. To find the normal component of acceleration $a_{\mathbf{N}}(t)$, we could use the similar formula

$$a_{\mathbf{N}}(t) = \mathbf{a}(t) \cdot \mathbf{N}(t),$$

but since we have already found $a_{\mathbf{T}}(t)$ it is easier to use the formula

$$a_{\mathbf{N}}(t) = \sqrt{||\mathbf{a}(t)||^2 - a_{\mathbf{T}}(t)^2}.$$

To apply the formula we first first find $||\mathbf{a}(t)||^2$:

$$||\mathbf{a}(t)||^2 = 0^2 + (-\cos t)^2 + (-\sin t)^2 = 1.$$

Therefore,

$$a_{\mathbf{N}}(t) = \sqrt{||\mathbf{a}(t)||^2 - a_{\mathbf{T}}(t)^2} = \sqrt{1-0} = 1.$$

So, the normal component of acceleration is $a_{\mathbf{N}}(t) = 1$.

4

There are multiple ways to find the limit; here are two.

First method. Whenever you see the expression x^2+y^2, it is usually a good idea to try polar coordinates, since in polar coordinates $x^2+y^2 = r^2$ (where r is the radius). In polar coordinate, $x = r\cos\theta, y = r\sin\theta$, so the function becomes

$$\begin{aligned}\frac{xy}{\sqrt{x^2+y^2}} &= \frac{(r\cos\theta)(r\sin\theta)}{\sqrt{r^2}} \\ &= \frac{r^2\cos\theta\sin\theta}{r} \\ &= r\cos\theta\sin\theta.\end{aligned}$$

Now the condition $(x,y)\to(0,0)$ expressed in polar coordinates is $r\to 0$. Thus,

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{r\to 0}(r\cos\theta\sin\theta).$$

Now $\sin\theta$ and $\cos\theta$ are both bounded in absolute value by 1: $|\sin\theta| \le 1, |\cos\theta| \le 1$. Therefore

$$|r\cos\theta\sin\theta| = r|\cos\theta|\cdot|\sin\theta| \le r\cdot 1\cdot 1 = r.$$

Thus,

$$0 \le |r\cos\theta\sin\theta| \le r.$$

Therefore, by the squeeze theorem, as $r\to 0$, the function $r\cos\theta\sin\theta$ will go to 0:

$$\lim_{r\to 0}(r\cos\theta\sin\theta) = 0.$$

Thus,

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0.$$

Second Method. Observe that

$$|x| = \sqrt{x^2} \le \sqrt{x^2+y^2}$$

and

$$|y| = \sqrt{y^2} \le \sqrt{x^2+y^2}.$$

Hence,

$$|xy| = |x| \cdot |y| \le \sqrt{x^2+y^2}\sqrt{x^2+y^2} = x^2+y^2.$$

Therefore,

$$\left|\frac{xy}{\sqrt{x^2+y^2}}\right| = \frac{|xy|}{\sqrt{x^2+y^2}} \le \frac{x^2+y^2}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2},$$

so

$$0 \le \left|\frac{xy}{\sqrt{x^2+y^2}}\right| \le \sqrt{x^2+y^2}.$$

Since

$$\lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2} = \sqrt{0^2+0^2} = 0$$

(since $\sqrt{x^2+y^2}$ is continuous), it follows by the squeeze theorem that

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0,$$

as with the first method.

5

Remember that $r^2 = x^2 + y^2$, where r is the distance from the origin to the point (x, y). Therefore, if a function only depends on $x^2 + y^2$, its graph will have radial symmetry, meaning it will look the same when rotated around the vertical axis (z-axis) by any angle. Looking at the five graphs, we see that only the second and third graphs have this symmetry. Looking at the five functions, we see that only $g(x, y) = \cos(x^2 + y^2)$ (choice (A)) and $g(x, y) = (x^2 + y^2)^{1/4}$ (choice (C)) depend only on $x^2 + y^2$. Specifically, we can write

$$\cos(x^2+y^2) = \cos(r^2)$$

and

$$(x^2+y^2)^{1/4} = (r^2)^{1/4} = \sqrt{r}.$$

Therefore, (A) and (C) must be matched with graphs 2 and 3 in some order. Now $\cos(r^2)$ oscillates repeatedly between 1 and -1 as r goes from 0 to ∞; therefore its graph must be "going up and down" like a wave. Clearly, among graphs 2 and 3, only graph 2 has this behavior; therefore (A) must be matched with graph 2. Hence, by process of elimination, (C) must go with graph 3.

Alternatively, we could note that the graph of $\sqrt{r}$ must look like the graph of the ordinary square-root function rotated around the vertical axis; clearly, among graphs 2 and 3, only graph 3 looks like this. So graph 3 goes with (C), and by elimination graph 2 must go with (A).

Now we are left with three graphs and functions. For (E), note that if we fix one variable (say x) and let the other variable (y) vary, then the function $\cos(x)\cos(y)$ will oscillate repeatedly. Since this works for either variable, the graph of $\cos(x)\cos(y)$ must oscillate up and down in both the x direction and the y direction. Clearly the last graph (graph 5) is the only one of the remaining three graphs with this behavior, so it must go with (E).

Now we are left with the functions $g(x,y) = |x|^{1/2}$ and $g(x,y) = x+y^2$. For any fixed value of x, say $x = c$, the corresponding vertical trace of $g(x,y) = x + y^2$ in the plane $x = c$ is the graph of the function of y defined by $f(y) = c^2 + y^2$, which is a parabola. Among the remaining graphs (graphs 1 and 4), only graph 4 has vertical traces that look like parabolas†. Hence, (B) must go with graph 4. By elimination, (D) must go with graph 1.

Alternatively, we could note that for any fixed value of y, say $y = c$, the corresponding vertical trace of $g(x,y) = |x|^{1/2}$ in the plane $y = c$ is the graph of the function $f(x) = |x|^{1/2}$, which has a "cusp" at $x = 0$. Among graphs 1 and 4, only graph 1 has vertical traces which look like this, so it must go with (D).

Thus, the answer is:

- (A) $\leftrightarrow$ graph 2
- (B) $\leftrightarrow$ graph 4
- (C) $\leftrightarrow$ graph 3
- (D) $\leftrightarrow$ graph 1
- (E) $\leftrightarrow$ graph 5

†*Note.* It may seem like graph 1 has vertical traces that are parabolas. However, these are "sideways" parabolas, unlike the the graph of $f(y) = c^2 + y^2$, which is a parabola opening "upwards" (i.e. a parabola with a vertical axis), as in graph 4. Moreover, the vertical traces of graph 1 are not really full parabolas, but just "halves" of parabolas, as opposed to the graph of $f(y) = c^2 + y^2$, which is an ordinary, full parabola.