Math 31b : Integration and Infinite Series
Solutions to Midterm 2
Winter 2013

1. Compute the indefinite integral

$$\int \frac{x+1}{x^2+16}\,dx.$$

Solution. We decompose the integral into two parts:

$$\int \frac{x+1}{x^2+16}\,dx = \int \frac{x}{x^2+16}\,dx + \int \frac{1}{x^2+16}\,dx.$$

For the first summand we use the substitution $x^2 = u$ (so $2xdx = du$) to get

$$\int \frac{x}{x^2+16}\,dx = \int \frac{du/2}{u+16} = \frac{1}{2}\ln|u+16| + C = \frac{1}{2}\ln|x^2+16| + C.$$

For the second summand we use the trigonometric substitution $x = 4\tan\theta$ (so $dx = (4\sec^2\theta)d\theta$) to get

$$\int \frac{1}{x^2+16}\,dx = \int \frac{(4\sec^2\theta)d\theta}{16(\tan^2\theta+1)} = \int \frac{(4\sec^2\theta)d\theta}{16(\sec^2\theta)} = \int \frac{d\theta}{4} = \frac{\theta}{4} + C = \frac{\tan^{-1}(x/4)}{4} + C.$$

Therefore, the original integral equals

$$\frac{1}{2}\ln|x^2+16| + \frac{\tan^{-1}(x/4)}{4} + C.$$

2. This problem asks you to set up the numerical approximations for the integral

$$\int_{1/4}^{1} \frac{1}{x}\,dx.$$

You don't have to compute the answer, just leave it as a sum. For example, for the midpoint approximation M_3, it suffices to write:

$$M_3 = \Delta x\big(f(3/8) + f(5/8) + f(7/8)\big) = \frac{1}{4}\Big(\frac{8}{3} + \frac{8}{5} + \frac{8}{7}\Big).$$

Now do the same for:

(a) the trapezoidal approximation T_3;

(b) the approximation S_6 by Simpson's rule.

Solution.
(a)

$$T_3 = \frac{\Delta x}{2}\big(f(1/4) + 2f(2/4) + 2f(3/4) + f(1)\big) = \frac{1}{8}\Big(\frac{4}{1} + 2\cdot\frac{4}{2} + 2\cdot\frac{4}{3} + 1\Big).$$

(b)

$$\begin{aligned} S_3 &= \frac{\Delta x}{3}\big(f(1/4) + 4f(3/8) + 2f(2/4) + 4f(5/8) + 2f(3/4) + 4f(7/8) + f(1)\big) \\ &= \frac{1}{24}\Big(\frac{4}{1} + 4\cdot\frac{8}{3} + 2\cdot\frac{4}{2} + 4\cdot\frac{8}{5} + 2\cdot\frac{4}{3} + 4\cdot\frac{8}{7} + 1\Big). \end{aligned}$$

3. (a) Write down the third Taylor polynomial $T_3(x)$ centered at $x = 0$ for the function $f(x) = \cos x$.

(b) Find the error bound for the value of $\cos(0.1)$ approximated by $T_3(0.1)$.

Solution.
(a) We have $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$, with values at $a = 0$:

$$f(0) = 1,\ f'(0) = 0,\ f''(0) = -1,\ f'''(0) = 0.$$

Therefore,

$$T_3(x) = 1 - \frac{x^2}{2!}.$$

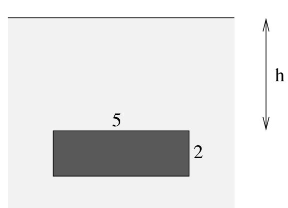
(b) The error bound is

$$|T_3(0.1) - f(0.1)| \le K_4\frac{|0.1 - 0|^4}{4!} = \frac{K_4}{24\cdot 10^4} = \frac{K_4}{240,000}.$$

Here, K_4 is the maximum of the absolute value of $f^{(4)}(x) = \cos x$ on $[0, 0.1]$, which is $\cos 0 = 1$. Thus the error is at most

$$\frac{1}{240,000}.$$

4. A rectangle of side lengths 5 (horizontally) and 2 (vertically) is submerged vertically in water such that its top edge is at depth h. For what value of h is the fluid force on the rectangle equal to $50\rho g$? (Here, ρ is the density of the water and g is the gravitational constant.)



Solution. The width is constant $f(y) = 5$. By the formula for the fluid force,

$$F = \rho g\int_h^{h+2} 5y\ dy = \rho g\cdot 5\frac{y^2}{2}\Big|_h^{h+2} = \rho g\cdot 5\frac{(h+2)^2 - h^2}{2} = \rho g\cdot 5\frac{4h+4}{2} = 10\rho g(h+1).$$

If $F = 50\rho g$, we must have $h + 1 = 5$ so $h = 4$.

5. Prove **from the definition** that

$$\lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0.$$

Solution. Given any $\epsilon > 0$, we need to find a number M such that, for all $n > M$,

$$\frac{1}{\sqrt{n}} < \epsilon$$

This is equivalent to $\sqrt{n} > 1/\epsilon$, or $n > 1/\epsilon^2$. So we can take $M = 1/\epsilon^2$.