Name: ____

UCLA ID Number: ______________________________________

Section letter: __________

Math 31b : Integration and Infinite Series

Final Exam

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You have 180 minutes.
No books, notes or calculators are allowed.
Do not use your own scratch paper.

1. Multiple Choice. (2 points each) Circle the correct answer. You do not need to justify your answer, and no partial credit will be given.

(i) The infinite series $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

$\boxed{(a)}$ converges to e^2

(b) converges to $e^{1/2}$

(c) converges to 2

(d) diverges to infinity

(e) None of the above.

(ii) The interval of convergence of the power series $\sum_{n=1}^{\infty} n(x-1)^n$ is

(a) $\{1\}$

(b) $(-1, 1)$

(c) $[-1, 1)$

$\boxed{(d)}$ $(0, 2)$

(e) $[0, 2)$

(f) None of the above.

(iii) If $\{a_n\}$ is a divergent sequence, then:

(a) $\{a_n\}$ is bounded

(b) $\{a_n\}$ is unbounded

(c) $\{a_n\}$ is monotonic

(d) $\{\frac{1}{a_n}\}$ converges

$\boxed{(e)}$ None of the above need to be true.

(iv) For $p > 1$, the alternating p-series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$

$\boxed{(a)}$ converges absolutely

(b) converges conditionally

(c) diverges to infinity

(d) diverges, but not to infinity

(v) If $\{a_n\}$ is a sequence of positive numbers such that $\lim_{n\to\infty} a_n = 0$, then

(a) $\sum_{n=1}^{\infty} (-1)^n a_n$ must converge absolutely

(b) $\sum_{n=1}^{\infty} (-1)^n a_n$ must converge conditionally

(c) $\sum_{n=1}^{\infty} (-1)^n a_n$ must converge, but we can't say if it does so absolutely or conditionally

(d) $\sum_{n=1}^{\infty} (-1)^n a_n$ must diverge

$\boxed{(e)}$ $\sum_{n=1}^{\infty} (-1)^n a_n$ may converge or diverge.

2. Multiple Choice. (2 points each) You do not need to justify your answer, and no partial credit will be given.

Write a letter (a-j) in each box, indicating the Maclaurin series that corresponds to the function $f(x)$:

(i) $f(x) = x \sin x$ $\boxed{h}$

(ii) $f(x) = x \cos x$ $\boxed{f}$

(iii) $f(x) = \tan^{-1}(x^2)$ $\boxed{e}$

(iv) $f(x) = \dfrac{1}{1+x^2}$ $\boxed{a}$

(v) $f(x) = \dfrac{1}{(1+x)^2}$ $\boxed{b}$

You can choose from:

(a) $\displaystyle\sum_{n=0}^{\infty}(-1)^n x^{2n}$;

(b) $\displaystyle\sum_{n=0}^{\infty}(-1)^n (n+1)x^n$;

(c) $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^n}{n!}$;

(d) $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+1}}{4n+1}$;

(e) $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+2}}{2n+1}$;

(f) $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n)!}$;

(g) $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+2)!}$;

(h) $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+2}}{(2n+1)!}$;

(i) $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+2}}{(2n+2)!}$;

(j) None of these.

3. Evaluate the following limits:

(a) (5 points)

$$\lim_{x \to 3} \frac{\sqrt{1+x}-2}{x-3}$$

(b) (5 points)

$$\lim_{x \to \infty} \left(1 - \frac{1}{x}\right)^{x^2}$$

Solution. (a) Using l'Hopital:

$$\lim_{x \to 3} \frac{\sqrt{1+x}-2}{x-3} = \lim_{x \to 3} \frac{\frac{1}{2\sqrt{1+x}}}{1} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

(b) Let $f(x) = (1 - 1/x)^{x^2}$. Then $\ln f(x) = x^2 \ln(1 - 1/x)$.

$$\lim_{x \to \infty} \ln f(x) = \lim_{x \to \infty} \frac{\ln(1 - \frac{1}{x})}{\frac{1}{x^2}} = \text{(by l'Hopital)} \lim_{x \to \infty} \frac{\frac{1/x^2}{1-\frac{1}{x}}}{-2/x^3} = \lim_{x \to \infty} -\frac{x-1}{2} = -\infty$$

so

$$\lim_{x \to \infty} f(x) = e^{\lim_{x \to \infty} lnf(x)} = e^{-\infty} = 0.$$

4. (10 points) Evaluate the integral

$$\int (\ln x)^2 \; dx$$

Solution. Substitute $u = \ln x$ so $x = e^u, dx = e^u du$. Then:

$$\int (\ln x)^2 \; dx = \int u^2 e^u du$$

Using integration by parts twice, we get

$$\int u^2 e^u du = u^2 e^u - \int 2ue^u du = u^2 e^u - 2ue^u + \int 2e^u du =$$
$$u^2 e^u - 2ue^u + 2e^u + C = (\ln x)^2 x - 2x \ln x + 2x + C.$$

5. (a) (5 points) Find the third Taylor polynomial $T_3(x)$ centered at 1 for the function $f(x) = \sqrt{x}$.

(b) (5 points) Estimate the error $|\sqrt{1.2} - T_3(1.2)|$. (You do not need to simplify your answer.)

Solution. (a) We have

$$f'(x) = (1/2)x^{-1/2},\ f''(x) = (1/2)(-1/2)x^{-3/2},\ f'''(x) = (1/2)(-1/2)(-3/2)x^{-5/2}$$

The values at $x = 1$ are

$$f(1) = 1,\ f'(1) = 1/2,\ f''(1) = -1/4,\ f'''(x) = 3/8.$$

Hence

$$\begin{aligned} T_3(x) &= 1 + \frac{1/2}{1!}(x-1) + \frac{-1/4}{2!}(x-1)^2 + \frac{3/8}{3!}(x-1)^3 \\ &= 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3. \end{aligned}$$

(b) $|f^{(4)}(x)| = |(1/2)(-1/2)(-3/2)(-5/2)x^{-7/2}| = |(15/16)x^{-7/2}|$. The maximum value of this on the interval $[1, 1.2]$ is $K_4 = 15/16$, so by the error bound for Taylor polynomials:

$$|\sqrt{1.2} - T_3(1.2)| \leq \frac{K_4(.2)^4}{4!} = \frac{15}{16 \cdot 24 \cdot 5^4}.$$

6. (10 points) Consider the half-circle S given by $x^2 + (y-4)^2 = 1$ and $y \geq 4$. Find the area of the surface of revolution obtained by rotating S around the x-axis.

Solution. We use the formula

$$S = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}dx$$

for $f(x) = 4 + \sqrt{1 - x^2}$ and $a = -1, b = 1$. Note that $f'(x) = -x/\sqrt{1 - x^2}$ so

$$\sqrt{1 + (f'(x))^2} = \sqrt{1 + \frac{x^2}{1 - x^2}} = \frac{1}{\sqrt{1 - x^2}}.$$

Therefore,

$$S = 2\pi \int_{-1}^{1} (4 + \sqrt{1 - x^2}) \cdot \frac{1}{\sqrt{1 - x^2}}dx = 2\pi(2 + 4\int_{-1}^{1} \frac{dx}{\sqrt{1 - x^2}})$$

Using the substitution $x = \sin\theta$ we get

$$S = 4\pi + 8\pi \int_{-\pi/2}^{\pi/2} \frac{\cos\theta}{\cos\theta} d\theta = 4\pi + 8\pi^2.$$

7. (10 points) Does the improper integral

$$\int_0^1 \frac{dx}{x^7 + x}$$

converge or diverge? Justify your answer carefully.

Solution. For $x \in (0,1)$ we have $x^7 \leq x$ so $x^7 + x \leq 2x$. Therefore,

$$\int_0^1 \frac{dx}{x^7 + x} \geq \int_0^1 \frac{dx}{2x}$$

The latter integral diverges by the p-test with $p = 1$. Hence, the original integral diverges as well, using the comparison test.

8. (5 points each) Do the infinite series

(a)

$$\sum_{n=1}^{\infty} \frac{1}{n \ln n + 1}$$

(b)

$$\sum_{n=1}^{\infty} \left(\frac{2n+\sqrt{n}}{n+2\sqrt{n}}\right)^n$$

converge or diverge? Justify your answers carefully.

Solution. (a) Use the limit comparison test with $a_n = 1/(n \ln n + 1)$ and $b_n = 1/(n \ln n)$. Since $\lim_{n\to\infty} a_n/b_n = 1$, it suffices to study the convergence of $\sum 1/(n \ln n)$. By the integral test, this is the same question as the convergence of the following improper integral, which can be evaluated using the substitution $e^u = x$:

$$\int_a^{\infty} \frac{dx}{x \ln x} = \int_{\ln a}^{\infty} \frac{e^u du}{e^u u} = \int_{\ln a}^{\infty} \frac{du}{u}$$

This diverges. Therefore, the series diverges.

(b) Use the Root Test:

$$\lim_{n\to\infty} \left(\frac{2n+\sqrt{n}}{n+2\sqrt{n}}\right) = 2 > 1$$

so the series diverges.

9. (5 points each) Evaluate the infinite series

(a)

$$\sum_{n=2}^{\infty} \frac{(-2)^n + 1}{3^n}$$

(b)

$$\sum_{n=2}^{\infty} \frac{1}{(n-1)(n+1)}$$

Solution. (a) Use the formula for geometric series:

$$\sum_{n=2}^{\infty} \frac{(-2)^n + 1}{3^n} = \Big(\frac{-2}{3}\Big)^2 \sum_{n=0}^{\infty} (-2/3)^n + \Big(\frac{1}{3}\Big)^2 \sum_{n=0}^{\infty} (1/3)^n = \frac{4}{9} \cdot \frac{1}{1 - \frac{-2}{3}} + \frac{1}{9} \cdot \frac{1}{1 - \frac{1}{3}} = \frac{4}{15} + \frac{1}{6} = \frac{13}{30}$$

(b) Use the partial fractions decomposition

$$\frac{1}{(n-1)(n+1)} = \frac{1}{2}\Big(\frac{1}{n-1} - \frac{1}{n+1}\Big)$$

and then telescoping:

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{1}{(n-1)(n+1)} &= \lim_{N\to\infty} \sum_{n=2}^{\infty} \frac{1}{(n-1)(n+1)} \\
&= \lim_{N\to\infty} \frac{1}{2}\Big(\frac{1}{1} - \cancel{\frac{1}{3}} + \frac{1}{2} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \cdots + \cancel{\frac{1}{N-2}} - \frac{1}{N} + \cancel{\frac{1}{N-1}} - \frac{1}{N+1}\Big) \\
&= \frac{1}{2} \lim_{N\to\infty} \Big(\frac{1}{1} + \frac{1}{2} - \frac{1}{N} - \frac{1}{N+1}\Big) \\
&= \frac{1}{2}\Big(\frac{1}{1} + \frac{1}{2}\Big) \\
&= \frac{3}{4}.
\end{aligned}$$

10. (a) (6 points) Find a power series $F(x) = \sum_{n=0}^{\infty} a_n x^n$ satisfying the differential equation $F'' = -F$, with initial conditions $F(0) = 1, F'(0) = 1$.

(b) (3 points) For what values of x does the series $F(x)$ converge? Justify your answer.

(c) (1 point) Write the function $F(x)$ in closed form—that is, as a simple expression such as $x\tan^{-1}(x), \cos(x^2)$, etc.

Solution.

(a) We have

$$\begin{aligned} F(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \ldots \\ F'(x) &= a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} + \ldots \\ F''(x) &= 2a_2 + 3 \cdot 2a_3 x + \cdots + n(n-1) a_n x^{n-2} + \ldots \end{aligned}$$

The initial conditions become $a_0 = F(0) = 1, a_1 = F'(0) = 1$. Equating coefficients in $F = -F''$ we get $a_{n-2} = -n(n-1)a_n$ so

$$a_0 = 1,\ a_2 = -\frac{1}{1 \cdot 2},\ a_4 = \frac{1}{1 \cdot 2 \cdot 3 \cdot 4}, \ldots, a_{2n} = (-1)^n \frac{1}{(2n)!}$$

$$a_1 = 1,\ a_3 = -\frac{1}{2 \cdot 3},\ a_5 = \frac{1}{2 \cdot 3 \cdot 4 \cdot 5}, \ldots, a_{2n+1} = (-1)^n \frac{1}{(2n+1)!}$$

We get

$$F(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

(b) The general term in $F(x)$ can be written as $\pm\frac{x^n}{n!}$. By the ratio test (where the sign doesn't matter since we take absolute values)

$$\lim_{n\to\infty} \frac{|x^{n+1}/(n+1)!|}{|x^n/n!|} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0$$

for all x, so $F(x)$ converges for all x.

(c) $F(x) = \sin x + \cos x$.