1. **[3 points]** Any cubic polynomial has:

   (a) At most 3 critical points.

   (b) Exactly 2 critical points.

   (c) At most 2 critical points.

   (d) Noe of the above.

   Solution. First note that since a cubic polynomial is differentiable everywhere, the only critical points will correspond to points at which the derivative is zero. Moreover, since our function is a cubic polynomial its derivative will have degree at most two. Thus the derivative will have at most two distinct zeros, and therefore the cubic polynomial has at most 2 critical points. So the correct answer is choice (c).

2. **[3 points]** The area of the triangle formed by the x-axis, y-axis, and the tangent line to the graph $y = (x+1)^{-2}$ at $x = 1$ is:

   $\left.\frac{dy}{dx}\right|_{x=1} = -2(x+1)^{-3}\Big|_{x=1} = -2\cdot 2^{-3} = \frac{-2}{8} = \frac{-1}{4}$ $\quad f(x) = \frac{1}{(x+1)^2}$ $\quad \frac{dy}{dx} = \frac{-2(x+1)}{(x+1)^4} = \frac{-4}{16} = \frac{-1}{4}$

   (a) 1

   (b) $\frac{1}{2}$

   (c) $\frac{1}{4}$

   (d) None of the above.

   $Y = \frac{-x}{4} + B$ $\quad Y(1) = \frac{1}{4}$

   $\frac{1}{4} = \frac{-(1)}{4} + B$ $\quad B = \frac{1}{2}$ $\quad Y = \frac{-x}{4} + \frac{1}{2}$ $\quad Y(0) = \frac{1}{2}$

   $0 = \frac{-x}{4} + \frac{1}{2}$ $\quad \frac{x}{4} = \frac{1}{2}$ $\quad x = 2$

   Solution. We first compute the equation for the tangent line:

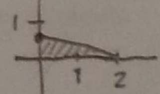
   $$y' = -2(x+1)^{-3}.$$

   Since $y(1) = (1+1)^{-2} = 2^{-2} = \frac{1}{4}$ and $y'(1) = -2(1+1)^{-3} = -2\cdot 2^{-3} = -\frac{1}{4}$ we know the tangent line has the following equation:

   $$L(x) = y'(1)(x-1) + y(1) = -\frac{1}{4}(x-1) + \frac{1}{4}.$$

   The triangle in question is a right triangle whose legs lie along the x and y axes, so its height is given by the y coordinate of the y-intercept and the its base-length is given by the x-coordinate of the x-intercept. The former quantity is simply $L(0) = -\frac{1}{4})(0-1) + \frac{1}{4} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. The latter is the number c where $L(c) = 0$. That is,

   $$\begin{aligned} 0 &= -\frac{1}{4}(c-1) + \frac{1}{4} \\ 0 &= -(c-1) + 1 \\ c - 1 &= 1 \\ c &= 2. \end{aligned}$$

   $A = \frac{1}{2}\left(\frac{1}{2}\right)(2) = \boxed{\frac{1}{2}}$

   Thus the triangle has area $\frac{1}{2}L(0)\cdot c = \frac{1}{2}\frac{1}{2}2 = \frac{1}{2}$, giving choice (b).

3. **[3 points]** Recall that a *real root* of a function $f(x)$ is a real number x_0 for which $f(x_0) = 0$. The function $f(x) = x^3 + 9x - 4$ has:

   $x^3 + 9x - 4 = 0$

   (a) exactly 1 real root.

   (b) exactly 2 real roots.

   (c) exactly 3 real roots.

   (d) None of the above.

Solution. Since $f(0) = 0 + 0 - 4 = -4 < 0$ and $f(1) = 1 + 9 - 4 = 6 > 0$, $f(x)$ has at least one real root somewhere in the interval $(0, 1)$. Now, suppose (towards a contradiction) that $f(x)$ has two or more distinct real roots. Say $x_1 < x_2$ are real roots of $f(x)$. Then since f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2) (in fact its is continuous and differentiable everywhere) and since $f(x_1) = 0 = f(x_2)$, Rolle's theorem tells us there should be a point c in (x_1, x_2) at which the derivative is zero. However, $f'(x) = 3x^2 + 9$ which is never zero and so we have a contradiction. Thus it must be that $f(x)$ has exactly one real root, choice (a).

4. **[3 points]** The Mean Value Theorem ensures the existence of some c in an interval (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

assuming that:

$\sqrt{1-x^2}\,dx$

(a) f(x) is continuous on the closed interval $[a, b]$.
(b) f(x) is differentiable on the open interval (a, b).
(c) f(x) is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b).
(d) None of the above.

Solution. The correct answer (as your text will tell you) is choice (c).

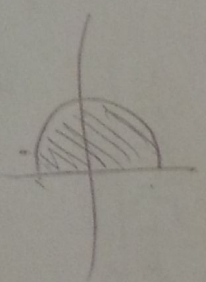
5. **[3 points]** The width of a rectangle is half its length. At what rate is the area of the rectangle increasing when its width is 10 cm and is increasing at $\frac{1}{2}$ cm/s?

(a) 5 cm^2/s
(b) 10 cm^2/s
(c) 20 cm^2/s
(d) None of the above.

$A = wl \quad l = 2w$

$A = 2w^2$

$\frac{dA}{dt} = 4w\left(\frac{dw}{dt}\right) = 4(10cm)\left(\frac{1}{2}cm/s\right) = \boxed{20cm^2/s}$



Solution. The area of a rectangle is $A = wl$ where w denotes width and l denotes length. Since we are told the width is half its length, $w = \frac{1}{2}l$ or $l = 2w$, the equation becomes $A = 2w^2$. Differentiating with respect to time t we have

$$\frac{dA}{dt} = 4w\frac{dw}{dt}.$$

We are asked to compute $\frac{dA}{dt}$ when $w = 10\ cm$ and $\frac{dw}{dt} = \frac{1}{2}\ cm/s$, and so we obtain

$$\frac{dA}{dt} = 4 \cdot 10\ cm \cdot \frac{1}{2}\ cm/s = 20\ cm^2/s,$$

or choice (c).

6. **[10 points]** Trader Joe's sells bananas for 19 cents each. Let N be the number of bananas sold let p be the price of a single banana. If Trader Joe's sells 300 bananas in a given day, and $\frac{dN}{dp} = -10$, estimate the number of bananas sold if the price for a banana is increased to 20 cents. Based on your estimate, would you recommend that they raise the price? Explain your answer.

Solution. We first produce a linear approximation for $N(p)$: the number of bananas sold per day in terms of price. Recall that a linear approximation to a function $f(x)$ at the point $x = a$ is of the form $\boxed{L(x) = f'(a)(x - a) + f(a)}$. Adapting this to our situation, our linear approximation would be

$$L(p) = \frac{dN}{dp}(p - 19) + N(19) = -10(p - 19) + 300.$$

Using this, we estimate that if the price per banana was 20 cents then the number of bananas Trader Joe's would sell is

$$L(20) = -10(20 - 19) + 300 = -10(1) + 300 = 290.$$

$L(p) = \frac{dN}{dp}(p-19) + N(19) = -10(p-19) + 300$

$L(20) = -10(20-19) + 300 = \boxed{290}$

To determine whether or not it is a good idea to make the price change we consider the revenue gained from each situation. Currently they sell 300 bananas at 19 cents a piece, yielding a total revenue of $300 \cdot .19 = 57$ dollars. After they change the revenue is $290 \cdot .20 = 58$ dollars. So raising the price would earn them more money and is recommended.

$(290)(0.20) = \$58$
$(300)(0.19) = \$57$

7. **[10 points]** Consider a box with a square bottom and no top. The material for the sides of the box is 2.5 cents per square centimeter, and the material for the bottom of the box is 13.5 cents per square centimeter. If the volume of the box is to be 100 cubic centimeters, find the dimension of the box that minimizes the total cost of materials. What is the minimum cost.

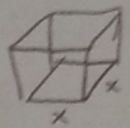
Solution. Let x denote the side length of the base of the box and y the height. Then base has area x^2 and the four sides have total area $4xy$. Hence the total cost is given by $T = 2.5 \cdot 4xy + 13.5x^2 = 10xy + 13.5x^2$ (cents). Our constraint is that the volume is 100 cubic centimeters, and so $100 = x^2y$. Solving the constraint for y (so that we may eliminate it from our equation for T) we have $y = 100/x^2$ and so

$$T = 10x\frac{100}{x^2} + 13.5x^2 = \frac{1000}{x} + 13.5x^2.$$

$C = A_b(13.5) + 4A_s(2.5)$

$y = \frac{100}{x^2}$

Next we have



$$\frac{dT}{dx} = -\frac{1000}{x^2} + 27x.$$

Setting this equal to zero we find that

① simplify to 1 variable
② derivative
③ set to zero
④ verify min/max

open interval
↳ in principle

$$\frac{1000}{x^2} = 27x$$
$$\frac{1000}{27} = x^3$$
$$\frac{10}{3} = x.$$

We need to verify that $x = \frac{10}{3}$ is a local minimum and we can use either the first or the second derivative test. Using the former we see that

$$\left.\frac{dT}{dx}\right|_{x=3} = -\frac{1000}{3^2} + 27 \cdot 3 = -\frac{1000}{9} + 81 = \frac{-1000 + 729}{9} = -\frac{271}{9} < 0$$
$$\left.\frac{dT}{dx}\right|_{x=4} = -\frac{1000}{4^2} + 27 \cdot 4 = -\frac{1000}{16} + 108 = \frac{-1000 + 1728}{16} = \frac{728}{16} > 0,$$

and so $x = \frac{10}{3}$ is indeed a local minimum. If we instead wanted to use the second derivative test we would first compute

$$\frac{d^2T}{dx^2} = \frac{2000}{x^3} + 27,$$

and then note that this is positive at $x = \frac{10}{3}$ so it is a local minimum.

Next we need to compute y; recall

$$y = \frac{100}{x^2} = \frac{100}{100/9} = 9.$$

$\frac{100}{\left(\frac{100}{9}\right)}$ $\quad \frac{100}{1}\left(\frac{9}{100}\right)$

$\frac{-6x - 4x^3 + 2x^5}{(1+x^2)^4}$

Lastly, the minimum total cost is

$$T = \frac{1000}{x} + 13.5x^2 = \frac{1000}{10/3} + 13.5\frac{100}{9} = 300 + \frac{1350}{9} = \frac{2700 + 1350}{9} = \frac{4050}{9} = 450 \text{ (cents)}.$$

inflection pts: $2x(x^2-3) = 0$; $x = 0$; $x = \pm\sqrt{3}$

$\frac{2x(1+x^2)[-(1+x^2) - 2(1-x^2)]}{(1+x^2)^{4}} = \boxed{\frac{2x(x^2-3)}{(1+x^2)^3}}$

8. **[15 points]** Showing all your work, provide a detailed sketch of the graph of the function

$$y = \frac{x}{1 + x^2}.$$

$\frac{dy}{dx} = \frac{(1+x^2) - x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$

denominator will never $= 0$ for all values of x

$1 - x^2 = 0$
$x^2 = 1$
$x = \pm 1$

$\frac{d^2y}{dx^2} = \frac{-2x(1+x^2)^2 - (1-x^2)[2(1+x^2)(2x)]}{(1+x^2)^{4}}$

$\frac{-2x(1+2x^2+x^4) - (1-x^2)(4x+4x^3)}{(1+x^2)^4} = \frac{(-2x - 4x^3 - 2x^5) - (4x - 4x^5)}{(1+x^2)^4}$

Solution. First note that since y is a rational function, it is continuous everywhere its denominator is non-zero. Since x^2+1 is always non-zero we then know that y is continuous for all real numbers and in particular it has no vertical asymptotes. Since the degree of the numerator is strictly smaller that the degree of the denominator we know that we have a horizontal asymptote at $y=0$. Also, y is zero precisely when the numerator x is zero; that is, the only zero is the point $(0,0)$. Next we determine the critical points:

$$y' = \frac{(1+x^2)\cdot 1 - x\cdot 2x}{(1+x^2)^2} = \frac{1+x^2-2x^2}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}.$$

Since the denominator $(1+x^2)^2$ is always positive there are no places where the derivative doesn't exist. Consequently the only critical points correspond to where $y'=0$, which in this case is when $1-x^2=0$. Hence our critical points are $x=\pm 1$. We'll hold off on determining whether these are local minima or maxima until after we have computed the second derivative, which we need to compute so that we can find the inflection points.

$$\begin{aligned} y'' &= \frac{(1+x^2)^2(-2x)-(1-x^2)2(1+x^2)(2x)}{(1+x^2)^4} = \frac{2x(1+x^2)[-(1+x^2)-2(1-x^2)]}{(1+x^2)^4} \\ &= \frac{2x[-1-x^2-2+2x^2]}{(1+x^2)^3} = \frac{2x(x^2-3)}{(1+x^2)^3} \end{aligned}$$

So we can see that $y''(-1) = \frac{-2(1-3)}{(1+1)^3} = \frac{4}{8} = \frac{1}{2} > 0$ and so $x=-1$ is a local minimum. Similarly, $y''(1) = \frac{2(1-3)}{(1+1)^2} = -\frac{1}{2}$ meaning $x=1$ is a local maximum. Given the numerator $2x(x^2-3)$, we see that $y''=0$ at $x=0,\pm\sqrt{3}$ and it is easy to see (by checking points) that the sign of y'' does change at each of these values. Thus the inflection points are $x=0,\pm\sqrt{3}$. By testing points in each of the appropriate intervals intervals we can gather the following data:

	$(-\infty,-\sqrt{3})$	$(-\sqrt{3},-1)$	$(-1,0)$	$(0,1)$	$(1,\sqrt{3})$	$(\sqrt{3},\infty)$
y'	-	-	+	+	-	-
y''	-	+	+	-	-	+

Lastly, we note $y(-\sqrt{3}) = -\frac{\sqrt{3}}{4}$, $y(-1) = -\frac{1}{2}$, $y(0)=0$, $y(1)=\frac{1}{2}$, $y(\sqrt{3}) = \frac{\sqrt{3}}{4}$. Putting all of this together leads us to the following picture: