1. **[3 points]** The derivative of $\frac{(x^2-x)^2}{x^3}$ is:
   (a) $1 - \frac{1}{x^2}$
   (b) $\frac{1}{x^3}(-3x^2 + 8x^2 - x + 3)$
   (c) [illegible] $+ \frac{1}{x^2}$
   (d) None of the above.

$$\frac{2x^3(2x^3-2x^2-x^2+x) - (x^2-x)^2(3x^2)}{x^6} = 1 - \frac{1}{x^2}$$

Solution. We first apply the quotient rule

$$\frac{d}{dx}\left(\frac{(x^2-x)^2}{x^3}\right) = \frac{x^3\frac{d}{dx}[(x^2-x)^2] - (x^2-x)^2\frac{d}{dx}(x^3)}{x^6} = \frac{x^3 2(x^2-x)(2x-1) - (x^2-x)^2 3x^2}{x^6},$$

where we have used the chain rule to say $\frac{d}{dx}[(x^2-x)^2] = 2(x^2-x)(2x-1)$. Next we expand and simplify the numerator

$$\begin{aligned}\frac{d}{dx}\left(\frac{(x^2-x)^2}{x^3}\right) &= \frac{2x^3(2x^3 - x^2 - 2x^2 + x) - (x^4 - 2x^3 + x^2)3x^2}{x^6} \\ &= \frac{2x^3(2x^3 - 3x^2 + x) - 3x^2(x^4 - 2x^3 + x^2)}{x^6} = \frac{4x^6 - 6x^5 + 2x^4 - 3x^6 + 6x^5 - 3x^4}{x^6} \\ &= \frac{x^6 - x^4}{x^6} = \frac{x^6}{x^6} - \frac{x^4}{x^6} = 1 - \frac{1}{x^2},\end{aligned}$$

and we see that this is choice (a).

2. **[3 points]** The derivative of $\cos^2(t^2)\tan(t^2)$ is:
   (a) $2t\cos(t^2)\tan(t^2) + 1$
   (b) $2t(\cos(t^2)\tan(t^2) + 1)$
   (c) $\cos(t^2)\tan(t^2) + 1)$
   (d) None of the above.

$$-2\cos(t^2)\sin(t^2)\cdot(2t)\tan(t^2) + \cos^2(t^2)\cdot\sec^2(t^2)(2t)$$

$$-\frac{4t\sin(t^2)}{\cancel{\cos(t^2)}}\cdot\cancel{\cos(t^2)}\cdot\sin(t^2) + \frac{\cancel{\cos^2(t^2)}}{\cancel{\cos^2(t^2)}}\cdot(2t)$$

$$= -4t\sin^2(t^2) + 2t$$

Solution. We first apply the product rule and chain rule $= \boxed{2t - 4t\sin^2(t^2)}$

$$\begin{aligned}\frac{d}{dt}\left(\cos^2(t^2)\tan(t^2)\right) &= \tan(t^2)\frac{d}{dt}\cos^2(t^2) + \cos^2(t^2)\frac{d}{dt}\tan(t^2) \\ &= \tan(t^2)2\cos(t^2)(-\sin(t^2))2t + \cos^2(t^2)\sec^2(t^2)2t.\end{aligned}$$

Next, recall that $\tan(t^2) = \frac{\sin(t^2)}{\cos(t^2)}$ and $\sec^2(t^2) = \frac{1}{\cos^2(t^2)}$. Thus

$$\frac{d}{dt}\left(\cos^2(t^2)\tan(t^2)\right) = 2\sin(t^2)(-\sin(t^2))2t + 2t = 2t - 4t\sin^2(t^2).$$

As this does not match any of the choices in (a)-(c) we see the answer is (d). You may have started the computation by noting $\cos^2(t^2)\tan(t^2) = \cos(t^2)\sin(t^2)$. If so, your computation would have led you to $2t\cos^2(t^2) - 2t\sin^2(t^2)$, which is the same as the result above via the trigonometric identity $\cos^2(\theta) = 1 - \sin^2(\theta)$.

$\pi x^{\pi-1}$

3. **[3 points]** Suppose that $F(x) = x^\pi$ and $G'(x) = \pi^2 x^{\pi-2}$. Then:
   (a) G is the derivative of $\frac{dF}{dx}$. $\pi a^{\pi-1}x$ $\pi(\pi-1)x^{\pi-2}$
   (b) A line tangent to F at a point a takes the form $y = \pi^2 a^{pi-2}x + b$. $(\pi^2-\pi)x^{\pi-2}$
   (c) A line tangent to G at a point a takes the form $y = \pi a^{\pi-1}x + b$.
   (d) None of the above. $\pi^2 a^{\pi-2}x$

Solution. Given the available choices, we are forced to consider each one. For choice (a), if G is the derivative of $\frac{dF}{dx} = F'$ then $G = F''$. Taking the derivative of each side of this we would get $G' = F^{(3)}$. We are given G' so to check whether or not this is true we need to compute $F^{(3)}$. Using the power rule successively we get

$$
\begin{aligned}
F'(x) &= \pi x^{\pi-1}\\
F''(x) &= \pi(\pi-1)x^{\pi-2}\\
F^{(3)}(x) &= \pi(\pi-1)(\pi-2)x^{\pi-3},
\end{aligned}
$$

so in fact $G' \neq F^{(3)}$ and the answer cannot be (a). For choice (b), recall that $F'(a)$ would give the slope of the tangent line at a point a. By our above computation we see that $F'(a) = \pi a^{\pi-1}$, which differs from the slope of the line given. So the answer is not (b). For choice (c), we again know that the tangent line would have slope $G'(a) = \pi^2 a^{\pi-2}$. Since this differs from the slope of the line given we know the answer cannot be (c) either. Hence the answer is (d).

4. **[3 points]** According to the definition of the derivative, the slope of a tangent line to a function $g(u)$ at a point r is described by:

   (a) $\frac{g(r+h)-g(r)}{h}$

   (b) $\lim_{h\to 0} \frac{g(r+h)+g(r)}{h}$

   (c) $\lim_{h\to 0} \frac{g(r+k)-g(r)}{k}$

   (d) None of the above.

Solution. Recall that the correct definition is

$$\lim_{h\to 0} \frac{g(r+h)-g(r)}{h}.$$

✓

So choice (a) is incorrect because it involves no limit. Choice (b) is incorrect because there is a sum instead of a difference in the numerator. Choice (c) is incorrect because the limit is with respect to h while the difference quotient uses k. This leaves us with (d) as the correct answer.

5. **[3 points]** Suppose that the water level in some lake is governed by

$$L(t) = \sin(t)\cos(t) + A\sin^2(\pi t)$$

for some constant A, where L is measured in inches (in) and t is measured in hours (h). How fast is the water level rising at $t = 0$?

   (a) 1 in

   (b) 0 in/h

   (c) 1 in/h

   (d) None of the above.

$\cancel{\cos(t)\sin(t)} + 2A\sin(\pi t)\cdot\cos(\pi t)\cdot(\pi)$

$\cos(t)\cos(t) - \sin(t)\sin(t)$

$1 - 0 + (2A)(0)(1)(\pi) = \boxed{1 \text{ in/h}}$

Solution. The question is asking us to compute $\frac{dL}{dt}\big|_{t=0}$. So differentiating both sides of the given formula yields

$$
\begin{aligned}
\frac{dL}{dt} &= \frac{d}{dt}\left(\sin(t)\cos(t) + A\sin^2(\pi t)\right) = \frac{d}{dt}\sin(t)\cos(t) + \frac{d}{dt}A\sin^2(\pi t)\\
&= \cos^2(t) - \sin^2(t) + A2\sin(\pi t)\cos(\pi t)\pi.
\end{aligned}
$$

Plugging in $t = 0$ gives $\frac{dL}{dt}\big|_{t=0} = 1 - 0 + A2\cdot 0\cdot 1\cdot \pi = 1$. Lastly, realize that dL/dt represents the change in water level over time, hence its units should be in/h and so the correct answer is (c).

6. **[10 points]** Find the tangent line to the curve

$$\frac{x^2}{y} + y(x-1) = \cos(\pi y)$$

$xy - y$

$$\frac{y(2x)-x^2(\frac{dy}{dx})}{y^2} + \frac{dy}{dx}(x-1) + Y = -\sin(\pi y)\cdot(\pi)\left(\frac{dy}{dx}\right)$$

at the point $(x, y) = (0, 1)$.

$0 + \frac{dy}{dx}(0-1) + 1 = 0$ $\quad \frac{dy}{dx} = 1$

Solution. We must first implicit differentiate and the determine $\frac{dy}{dx}$ when $x = 0$ and $y = 1$. On the left hand side of the equation we'll use the quotient and product rules and on the right we'll use the chain rule:

$$\frac{d}{dx}\left(\frac{x^2}{y} + y(x-1)\right) = \frac{d}{dx}\cos(\pi y)$$

$$\frac{y(2x) - x^2\frac{dy}{dx}}{y^2} + \frac{dy}{dx}(x-y) + y = -\pi\sin(\pi y)\frac{dy}{dx}.$$

$Y = mx + b$

$1 = (1)(0) + b$

$b = 1$

$\boxed{Y = X + 1}$

At this point, a lot of people begin to solve for $\frac{dy}{dx}$ *before* substituting $x = 0$ and $y = 1$ into the equation. This involves a lot more work and may not be valid if you divide by x at any point (as it will equal zero after our substitution). If you instead substitute $x = 0$ and $y = 1$ the above equation becomes

$$0 + \frac{dy}{dx}(0-1) + 1 = 0.$$

A lot of people were confused about the value of $\sin(\pi) = 0$. Remember that sine is zero at all integer values of π, whereas cosine alternates between ± 1. Now, this equation reduces to $\frac{dy}{dx} = 1$, and so we see that our tangent line is going to have slope $m = 1$. Using the slope-intercept form for a line $y = mx + b$ we solve for b by using $(x, y) = (0, 1)$:

$$1 = 1 \cdot 0 + b,$$

or $b = 1$. Thus the tangent line at $(x, y) = (0, 1)$ has the equation $y = x + 1$.

7. **[10 points]** Air is pumped into a spherical balloon at a rate of 20 cm^3/min. Determine the rate at which the radius is changing when the balloon diameter is 10 cm. It may be helpful to remember that the volume of a sphere is described by $\frac{4}{3}\pi r^3$.

$\frac{dV}{dt} = 20 cm^3/min \qquad \frac{dr}{dt} = ?$

Solution. We immediately note that since the *diameter* is 10 cm then the radius is 5 cm. Next we use the given equation to write $V(t) = \frac{4}{3}\pi r(t)^3$. Now, the information in the first sentence of the problem translates to $\frac{dV}{dt} = 20cm^3/min$, and we are asked to determine $\frac{dr}{dt}$ when $r = 5cm$. Since we already have equation relating V and r we simply need to differentiate both sides with respect to t and substitute in the relevant quantities:

$$\frac{dV}{dt} = 4\pi r^2\left(\frac{dr}{dt}\right)$$

$$20 cm^3/min = 4\pi(5^2)\left(\frac{dr}{dt}\right)$$

$$\boxed{\frac{1}{5\pi} cm/min}$$

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$\frac{dV}{dt} = \frac{4}{3}\pi 3r^2\frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}.$$

Substituting $dV/dt = 20$ and $r = 5$ gives

$$20 = 4\pi(5)^2\frac{dr}{dt},$$

or

$$\frac{dr}{dt} = \frac{20}{4\pi 5^2} = \frac{20}{100\pi} = \frac{1}{5\pi}.$$

8. **[15 points]** Suppose the semicircle in each of the three figures had radius 1, and recall that the shaded sector in the center has area $\frac{\theta}{2}$.

(a) Comparing the smaller of the two triangles and the sector, find a function bounding $\frac{\sin\theta}{\theta}$.

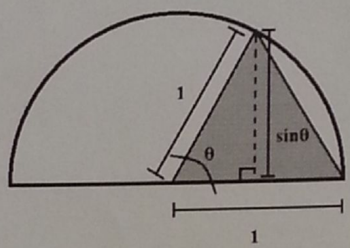
(b) Comparing the larger of the two triangles and the sector, find a second function bounding $\frac{\sin\theta}{\theta}$.

(c) Carefully state the squeeze law for limits.

(d) Making use of parts (a)-(c), prove that $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$. Justify all of your work.

(e) Using the definition of the derivative, prove that $\frac{d}{d\theta}\cos\theta=-\sin\theta$ (you may find the identity $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$ useful). For full points, you should justify all limits and limit laws used.

Solution. (a) We will use the fact that the smaller triangle has less area than the sector. The base of the smaller triangle is 1 (it is a radius of the semicircle) and the height of the circle is opposite the angle with measure θ in the right triangle with the radius 1 as its hypotenuse.



Hence the height is given by $\sin(\theta)$. Thus the smaller triangle has area

$$\frac{1}{2}1\cdot\sin\theta\leq\frac{\theta}{2},$$

which is the area of the sector. Multiplying the inequality by 2 and dividing by θ yields

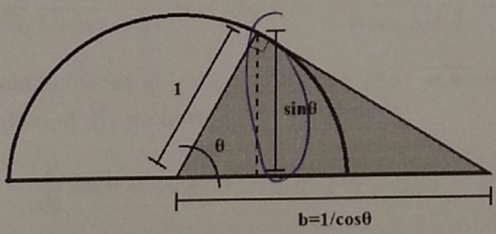
$$\frac{\sin\theta}{\theta}\leq 1.$$

(b) Here we use the fact that the area of the larger triangle is bigger than $\frac{\theta}{2}$, the area of the sector. The larger triangle has the same vertical height as the smaller one, that is $\sin(\theta)$. If the horizontal base, which is the hypotenuse of this triangle, has a length of b then $\cos\theta=\frac{1}{b}$ since the adjacent leg of the triangle is a radius of the semicircle and so has length one.

$\cos\theta=\frac{1}{b}$

$\cos\theta=\frac{1}{\frac{1}{\cos\theta}}$

$b=\frac{1}{\cos\theta}$



So, $b=\frac{1}{\cos\theta}$ and the larger triangle has area

$A=\frac{1}{2}bh$

$$\frac{1}{2}\sin\theta\frac{1}{\cos\theta}\geq\frac{\theta}{2}.$$

$A=\frac{1}{2}(\sin\theta)(\frac{1}{\cos\theta})\geq\frac{\theta}{2}$

Multiplying both sides by $2\cos\theta$ and dividing by θ yields

$$\frac{\sin\theta}{\theta}\geq\cos\theta.$$

$\frac{\sin\theta}{\theta}\geq\cos\theta$

(c) If squeeze law states that if the inequality $l(x) \leq f(x) \leq u(x)$ holds on an open interval (a, b) containing c (but not necessarily at $x = c$) and

$$\lim_{x\to c} l(x) = \lim_{x\to c} u(x) = L,$$

Then the limit of $f(x)$ as x approaches c exists and

$$\lim_{x\to c} f(x) = L.$$

(d) Note that $\lim_{\theta\to 0} 1 = 1$ as 1 is a constant function, and $\lim_{\theta\to 0}\cos\theta = \cos(0) = 1$ because cosine is a continuous function. Also, from parts (a) and (b) we know

$$\cos\theta \leq \frac{\sin\theta}{\theta} \leq 1.$$

Hence, by the squeeze law we have

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$$

(e) Keeping the aforementioned trigonometric identity in mind, recall the definition of the derivative:

$$\frac{d}{d\theta}\cos\theta = \lim_{h\to 0}\frac{\cos(\theta+h)-\cos\theta}{h} = \lim_{h\to 0}\frac{\cos\theta\cos h - \sin\theta\sin h - \cos\theta}{h}.$$

Next we apply the sum law and the constant multiple law to get

$$\frac{d}{d\theta}\cos\theta = \cos\theta\lim_{h\to 0}\frac{\cos h - 1}{h} - \sin\theta\lim_{h\to 0}\frac{\sin h}{h}.$$

The second limit on the right we know is 1 by part (d). Let us compute the other limit:

$$\lim_{h\to 0}\frac{\cos h - 1}{h} = \lim_{h\to 0}\frac{\cos h - 1}{h}\frac{\cos h + 1}{\cos h + 1} = \lim_{h\to 0}\frac{\cos^2 h - 1}{h(\cos h + 1)}.$$

Remember that $\sin^2 x + \cos^2 x = 1$, so $\cos^2 h - 1 = -\sin^2 h$. Using this, the product law, and again our result from part (d) we obtain

$$\lim_{h\to 0}\frac{\cos h - 1}{h} = \lim_{h\to 0}\frac{-\sin^2 h}{h(\cos h+1)} = -\lim_{h\to 0}\frac{\sin h}{h}\lim_{h\to 0}\frac{\sin h}{\cos h + 1} = -\lim_{h\to 0}\frac{\sin h}{\cos h + 1}.$$

The function $\frac{\sin x}{\cos x + 1}$ is continuous at $x = 0$, and so the limit is simply $\frac{\sin(0)}{\cos(0)+1} = \frac{0}{0+1} = 0$. Returning to our computation of $\frac{d}{d\theta}$, we have

$$\frac{d}{d\theta}\cos\theta = \cos\theta\cdot 0 - \sin\theta\cdot 1 = -\sin\theta.$$

~~cos(2x) = sin²x − 1~~

$\cos^2(x) + \sin^2(x) = \underline{1}$