Midterm 1, Math 31A, Fall 2013
Instructor: Tonći Antunović

Printed name a

Signed name: _

Section number

Instructions:

- Read problems very carefully. Please raise your hand if you have questions at any time.
- The correct final answer alone is not sufficient for full credit - try to explain your answers as much as you can. This way you are minimizing the chance of losing points for not explaining details, and maximizing the chance of getting partial credit if you fail to solve the problem completely. However, try to save enough time to seriously attempt to solve all the problems.
- If it's obvious that you final answers can be simplified, please simplify them. Otherwise, your final answers need to be simplified only if this is required in the statement of the problem.
- You are allowed to use only the items necessary for writing. Notes, books, sheets, calculators, phones are not allowed, please put them away. If you need more paper please raise your hand (you can write on the back pages).

Question	Points	Score
1	10	10
2	10	6
3	10	3
4	10	10
5	10	6
Total:	50	35

70%

10

1. (a) (4 points) Write down the proof of the power rule:

$$\frac{d}{dx}x^n = nx^{n-1}.$$

$f(x) = nx^{n-1}$

$f'(a) = \lim_{x\to a} \frac{x^n - a^n}{x-a}$

$(x^2 + a^2) = (x-a)(x+a)$

$(x^3 + a^3) = (x-a)(x^2+xa+a^2)$

$(x^4 + a^4) = (x-a)(x^3+x^2a+ax^2+a^3)$

generalization:

$(x^n \; a^n) = (x-a)(x^{n-1}+x^{n-2}a+x^{n-3}a^2+\ldots+xa^{n-2}+a^{n-1})$

$f'(a) = \lim_{x\to a} \frac{(x-a)(x^{n-1}+x^{n-2}a+x^{n-3}a^2+\ldots+xa^{n-2}+a^{n-1})}{x-a}$

$= \lim_{x\to a} (x^{n-1}+x^{n-2}a+x^{n-3}a^2+\ldots x^{n-2}+a^{n-1})$

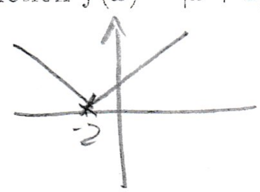
$= (a^{n-1}+a^{n-2}a+a^{n-3}a^2+\ldots+a^{n-2}+a^{n-1})$ [n terms]

$= n\cancel{x}a^{n-1}$

4

(b) (2 points) Find all points x where the function $f(x) = |x+2|$ is not differentiable.

$x = -2$



2

(c) (2 points) Compute the 15th derivative of $\cos x$.

$f(x) = \cos x$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

$f''''(x) = \cos x$

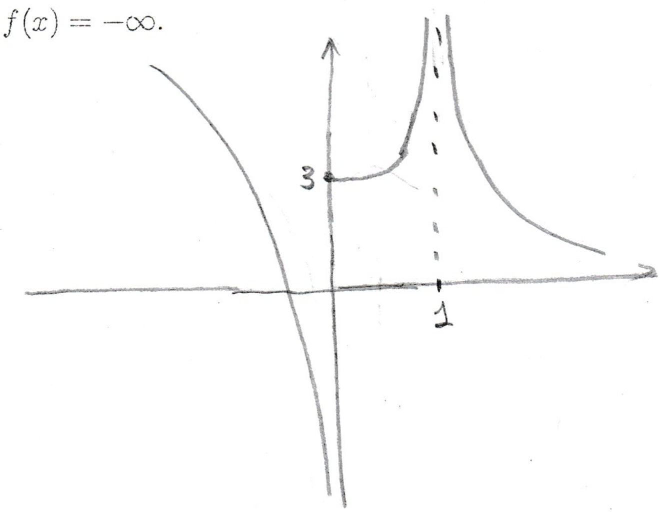
derivative repeats every 4 times

$f^{(15)}(\cos x) = \sin x$

2

(d) (2 points) Sketch the graph of a function f which satisfies each of the conditions

- $\lim_{x\to 1} f(x) = +\infty$,
- $f(0) = 3$ and f is right-continuous at $x = 0$,
- $\lim_{x\to 0-} f(x) = -\infty$.



2

2. Compute the limits

(a) (5 points)

$$\lim_{t\to 0}\frac{\sqrt{t^4+t^2+1}-\sqrt{t^2+1}}{t^4}.$$

$$\frac{\sqrt{t^4+t^2+1}-\sqrt{t^2+1}}{t^4}\cdot\frac{\sqrt{t^4+t^2+1}+\sqrt{t^2+1}}{\sqrt{t^4+t^2+1}+\sqrt{t^2+1}}$$

5

$$=\frac{t^4+\cancel{t^2}+\cancel{1}-\cancel{(t^2+1)}}{t^4\left(\sqrt{t^4+t^2+1}+\sqrt{t^2+1}\right)}=\frac{\cancel{t^4}}{\cancel{t^4}\left(\sqrt{t^4+t^2+1}+\sqrt{t^2+1}\right)}=\frac{1}{\sqrt{t^4+t^2+1}+\sqrt{t^2+1}}$$

substitute 0 into limit $\frac{1}{\sqrt{1}+\sqrt{1}}=\frac{1}{2}$

(b) (5 points)

$$\lim_{t\to 0+}\frac{\sqrt{\sin t}}{\sqrt{\sin t}+\sqrt{t}}.$$

1

$$\frac{\sqrt{\sin t}}{\sqrt{\sin t}+\sqrt{t}}\cdot\frac{\sqrt{\sin t}-\sqrt{t}}{\sqrt{\sin t}-\sqrt{t}}=\frac{\sin t-\sqrt{t}\sqrt{\sin t}}{\sin t-t}$$

$$\frac{\cancel{\sin t}(1-\sqrt{t\sin t})}{\cancel{\sin t}(1-t)}=\frac{(1-\sqrt{t\sin t})}{1-t}$$

substitute 0 into limit

$$\lim_{t\to 0+}\frac{(1-\sqrt{0\sin(0)})}{1-0}=\frac{1-0}{1}=1$$

Divide both numerator & denominator by $\sqrt{t}$

Since $\lim_{t\to 0+}\frac{\sin t}{t}=1$

$$\lim_{t\to 0+}\frac{\sqrt{\frac{\sin t}{t}}}{\sqrt{\frac{\sin t}{t}}+\cancel{\sqrt{t}}\,1}\ \#\ \longrightarrow\ \frac{1}{1+1}=\frac{1}{2}$$

3. (10 points) Compute the one-sided limits

$$\lim_{t\to 1-} \frac{t^2-1}{(t-1)^2}, \quad \text{and} \quad \lim_{t\to 1+} \frac{t^2-1}{(t-1)^2}.$$

$\lim_{t\to 1-} \frac{t^2-1}{(t-1)^2}$ $t<1$

As $t\to 1-$, value of the function becomes infinitely large

$\therefore \lim_{t\to 1-} \frac{t^2-1}{(t-1)^2} = +\infty$

$\lim_{t\to 1+} \frac{t^2-1}{(t-1)^2}$ $t>1$

As $t\to 1+$, value of the function becomes infinitely small

$\therefore \lim_{t\to 1+} \frac{t^2-1}{(t-1)^2} = -\infty$

x	y
-1	0
0	-1
1	∞
2	3

y is indeterminate at x = 1

$\therefore$ limit is infinity

determine signs

$\lim_{t\to 1-} \frac{t^2-1}{(t-1)^2}$

$\lim_{t\to 1-} \frac{(t+1)\cancel{(t-1)}}{(t-1)\cancel{(t-1)}}$

$\lim_{t\to 1-} \frac{(t+1)}{(t-1)}$ | $\lim_{t\to 1+} \frac{(t+1)}{(t-1)}$

↳ on the right numerator on both cases converges to 2 and the denominator to zero

$t\to 1-$ denominator is negative

$t\to 1+$ denominator is positive

$\lim_{t\to 1-} = -\infty$

$\lim_{t\to 1+} = +\infty$

$-\infty$

$\cos t - (2t+3)^{-\frac{1}{2}}$

4. (10 points) Compute the second derivative of

10

$$f(t) = t\sin t + \cos t - \frac{1}{\sqrt{2t+3}}.$$

$f'(t) = (t \cdot \sin t' + \sin t \cdot t') + (\cos t)' - ((2t+3)^{-\frac{1}{2}})'$

$= (t\cos t + \sin t) + (-\sin t) - [-\frac{1}{2}(2t+3)^{-\frac{3}{2}} \cdot 2]$

$= t\cos t + (2t+3)^{-\frac{3}{2}}$

$f''(t) = (t \cdot \cos t' + \cos t \cdot t') + [(2t+3)^{-\frac{3}{2}}]'$

$= (-t\sin t + \cos t) + (-\frac{3}{2}(2t+3)^{-\frac{5}{2}} \cdot 2)$

$= \cos t - t\sin t + (-3(2t+3)^{-\frac{5}{2}})$

$= \cos t - t\sin t - 3(2t+3)^{-\frac{5}{2}}$

5. (10 points) Consider the function $f(x) = \frac{1}{(x-1)^2}$. Compute the equations of all tangent lines to the graph of f whose slope is equal to -2.

$f(x) = \frac{1}{(x-1)^2} = (x-1)^{-2}$

$f'(x) = -2(x-1)^{-3}$ ✓

Slope equal to -2 $\quad f'(x) = -2$

$-2 = -2(x-1)^{-3}$

$-2 = -2\left(\frac{1}{(x-1)^3}\right)$

$-2 = \frac{-2}{(x-1)^3}$

$+1 = \frac{1}{(x-1)^3}$

$-1(x-1)^3 = 1$

$-1(x^2-2x+1)(x-1) = 0$

$-2(x^2-2x+1)(x-1) = 0$

at these points slope $= -2$

① $x = -2 \quad y = \frac{1}{9}$

② $x = 1$

③ $x = -1 \quad y = \frac{1}{4}$

① $\frac{1}{9} = -2(-2) + c$

$\frac{1}{9} = 4 + c \quad \boxed{y = -2x - \frac{35}{9}}$

$1 = 36 + 9c$

$9c = -35$

$c = \frac{-35}{9}$

② $x = 1$ indeterminate

$\cancel{-2} = -\cancel{2}(x-1)^{-3}$

$\frac{1}{(x-1)^3} = 1 \quad \underline{x = 2} \quad y = \underline{1}$

equation of tangent line

$\boxed{y - 1 = -2(x-2)}$

Expand $(x-1)^3$

$(x-1)(x-1)(x-1)$

$= (x^2-2x+1)(x-1)$

$= x^3 - x^2 - 2x^2 + 2x + x - 1$

$= x^3 - 3x^2 + 3x - 1$

either $(x-1) = 0$

$(x^2-2x+1) = 0$

$(-2x^2-4x-2) = 0$

$2x^2+4x+2$

$2x \quad x \quad 1 \quad 2$

$2 \quad 1$

$(2x+2)(x+1) = 0$

③ $\frac{1}{4} = -2(-1) + c$

$\frac{1}{4} = 2 + c \quad \boxed{y = -2x - \frac{7}{4}}$

$1 = 8 + 4c$

$4c = -7$

$c = \frac{-7}{4}$