Trigonometric identities

$$\cos 2nx = \cos^2(nx) - \sin^2(nx)$$

$$\sin mx \cos nx = \tfrac{1}{2}\left[\sin(m+n)x + \sin(m-n)x\right]$$

$$\cos mx \cos nx = \tfrac{1}{2}\left[\cos(m+n)x + \cos(m-n)x\right]$$

$$\sin mx \sin nx = \tfrac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right]$$

Question 1 (*10 points*)

Given that $y_1(x) = x$ is one of the solutions to the following differential equation

$$x^2y'' + 2xy' - 2y = 0,$$

find the general solution, $y(x)$, in the form

$$y(x) = c_1y_1(x) + c_2y_2(x),$$

where c_1 and c_2 are constants.

$y_1 = x$, $\quad y(x) = C_1 x + C_2 y_2(x) \quad y_2 = V(x)\cdot x$, $\quad y_2' = V + V'x$

let $\quad y = C_1 x + C_2 V x \quad y' = C_1 + C_2 V' x + C_2 V \quad y_2'' = V' + V''x + V' = 2V' + V''x$

$y'' = C_2 V'' x + C_2 V' + C_2 V'$

plug in $\quad x^2 y'' + 2xy' - 2y = 0$

$$x^2[C_2 V'' x + C_2 V' + C_2 V'] + 2x[C_1 + C_2 V' x + C_2 V] - 2C_1 x - 2C_2 V x = 0$$

$$C_2 x^2 \cdot xV'' + C_2 V' x^2 + C_2 V' x^2 + 2xC_1 + 2C_2 V' x^2 + 2xC_2 V - 2C_1 x - 2C_2 V x = 0 \; (1)$$

Since $y_2(x)$ is one of solution, $\quad x^2 y_2'' + 2xy_2' - 2y_2 = 0$

$$x^2[2V' + V''x] + 2x[V + V'x] - 2Vx = 0 \; (2) \Rightarrow x^2[2V' + V''x] + 2x^2 V' = 0$$

I have (1) $\to$ (3):

$$C_2 x^2 \cdot xV'' + C_2 V' x^2 + C_2 V' x^2 + 2C_2 V' x^2 = 0 \; (3)$$

$$\Rightarrow C_2 x^2 (2V' + V''x) + 2C_2 V' x^2 = 0 \qquad P = \frac{2}{x}$$

using: $\quad y_2 = \int \frac{e^{-\int P dx}}{y_1^2} dx = \int \frac{e^{-\int \frac{2}{x} dx}}{x^2} dx$

$$y_2 = \int \frac{e^{\ln x^{-2}}}{x} dx = \int \frac{1}{x^3} dx = \frac{-1}{2x^2}$$

10

$\therefore y_2 = \frac{-1}{2x^2}$

$y_1 = x$

$\therefore y = C_1 y_1 + C_2 y_2$

$\therefore y = C_1 y_1 + C_2 y_2$ general solution

$$y_g = C_1 x + C_2 \cdot \left(\frac{-1}{2x^2}\right) = C_1 x - C_2 \frac{1}{2x^2}$$

Question 2 (*25 points*)

(a) (*8 points*) The Laplace transform of a function $f(x)$ is defined as

$$L[f(x)] = \int_0^\infty e^{-sx} f(x)\,dx; \tag{1}$$

using (1), show that

$$L[H_a(x)f(x-a)] = e^{-as}F(s),$$

where $H_a(x)$ is the shifted Heaviside function and $F(s) = L[f(x)]$.

(b) (*8 points*) A function $g(x)$ is defined by

$$g(x) = \begin{cases} x & x < 3 \\ 6 - x & x \geq 3 \end{cases}$$

Sketch the function $g(x)$ in $[0, 6]$ and write down $g(x)$ in terms of the shifted Heaviside function $H_3(x)$.

(c) (*9 points*) Show that the Laplace transform of $g(x)$ is given by,

$$G(s) = \frac{1 - 2e^{-3s}}{s^2}$$

$$H_a(x) = \begin{cases} 1 & x \geq a \\ 0 & x < a \end{cases}$$

---

a) $L[H_a(x)f(x-a)] = \int_0^\infty H_a(x) f(x-a) e^{-sx} dx$

$= \int_0^a H_a(x) e^{-sx} f(x-a) dx + \int_a^\infty H_a(x) e^{-sx} f(x-a) dx$

$= \int_a^\infty e^{-sx} f(x-a)\, dx,$ let $x - a = u$, $du = dx$

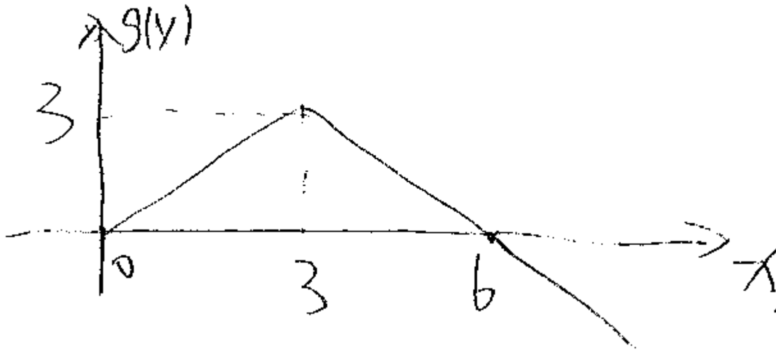
$x = a + u$, $u \in [0, +\infty)$

$L[H_a(x)f(x-a)] = \int_0^\infty e^{-s(a+u)} f(u) du = e^{-as} \int_0^\infty e^{-su} f(u) du = e^{-as} F(s)$

$L[H_a(x)f(x-a)] = e^{-as} F(s)$ ✓

8

b) $g(x) = \begin{cases} x & x<3 \\ 6-x & x\geq 3 \end{cases}$



$\therefore g(x) = [1-H_3(x)]\,x + H_3(x)(6-x)$

$g(x) = x - H_3(x)\cdot x + 6H_3(x) - xH_3(x) = x + 6H_3(x) - 2xH_3(x)$

$\Rightarrow g(x) = x + 2H_3(x)(3-x) = x - 2H_3(x)\cdot[x-3]$

c) $G(s) = L[g(x)] = L[x] - 2L[H_3(x)(x-3)]$

$$= \frac{1}{s^2} - 2\frac{e^{-3s}}{s^2} = \frac{1-2e^{-3s}}{s^2}$$

$$\therefore G(s) = \frac{1-2e^{-3s}}{s^2}$$

Question 3 (*15 points*)

Using the Laplace transform technique, solve the following IVP

$$y'' + y' = g(x), \quad y(0) = 1, y'(0) = 0,$$

where $g(x)$ is the function defined in Question 2(b).

$y'' + y' = g(y) \Rightarrow$ I take laplace transform:

$$[s^2 Y(s) - s y(0) - y'(0)] + [s Y(s) - y(0)] = G(s)$$

$$s^2 Y(s) - s - 0 + s Y(s) - 1 = \frac{1 - 2e^{-3s}}{s^2}$$

$$(s^2 + s) Y(s) = \frac{1 - 2e^{-3s}}{s^2} + (s+1)$$

$$Y(s) = \frac{1 - 2e^{-3s}}{s^2(s^2+s)} + \frac{s+1}{s^2+s} = \frac{1 - 2e^{-3s}}{s^3(1+s)} + \frac{1}{s}$$

$$\frac{1}{s^3(1+s)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{1+s}$$

$$= \frac{As^2(1+s) + Bs(1+s) + C(1+s) + Ds^3}{s^3(1+s)}$$

$$= \frac{As^2 + As^3 + Bs + Bs^2 + C + Cs + Ds^3}{s^3(1+s)}$$

$$= \frac{(A+D)s^3 + s^2(B+A) + s(B+C) + C}{s^3(1+s)}$$

$$\Rightarrow \begin{cases} A+D=0 \\ A+B=0 \\ B+C=0 \\ C=1 \end{cases} \Rightarrow \begin{cases} A=-B=1 \\ B=-1=-C \\ C=1 \\ D=-A=-1 \end{cases}$$

$$\therefore \frac{1}{s^3(s+1)} = \frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3} - \frac{1}{s+1}$$

$$\therefore Y(s) = \left(\frac{2}{s} - \frac{1}{s^2} + \frac{1}{s^3} - \frac{1}{s+1}\right) - 2\left[\frac{e^{-3s}}{s} - \frac{e^{-3s}}{s^2} + \frac{e^{-3s}}{s^3} - \frac{e^{-3s}}{s+1}\right]$$

$$Y(s) = \left(\frac{2}{s} - \frac{1}{s^2} + \frac{2!}{2s^3} - \frac{1}{s+1}\right) - 2\left[\frac{e^{-3s}}{s} - \frac{e^{-3s}}{s^2} + \frac{1}{2}\cdot\frac{2e^{-3s}}{s^3} - \frac{e^{-3s}}{s+1}\right]$$

$$\Rightarrow L^{-1}[Y(s)] = y(x)$$

$$y(x) = 2 - x + \frac{1}{2}x^2 - e^{-x} - 2H_3(x)\left[1 - (x-3) + \frac{1}{2}(x-3)^2 - e^{-(x-3)}\right]$$

$$\therefore y(x) = 2 - x + \frac{x^2}{2} - e^{-x} - 2H_3(x)\left[1 - (x-3) + \frac{1}{2}(x-3)^2 - e^{-(x-3)}\right]$$

Question 4 (*30 points*)

(a) Consider an *even* function $f(x)$ with period $T = \pi$. Its Fourier series representation is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx). \tag{1}$$

Starting from Eq. (1), derive the following Fourier coefficient formulas for a_n,

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx)\, dx, \qquad n \geq 0. \tag{2}$$

Consider the function $f(x) = |\sin x|$ with period $T = \pi$.

(b) Show that the function is *even* and sketch $f(x)$ in $-\pi \leq x \leq 3\pi$.

(c) By making use of Eqs. (1) and (2), show that the Fourier series of $f(x) = |\sin x|$ is given by

$$f(x) \sim \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \cos(2nx).$$

---

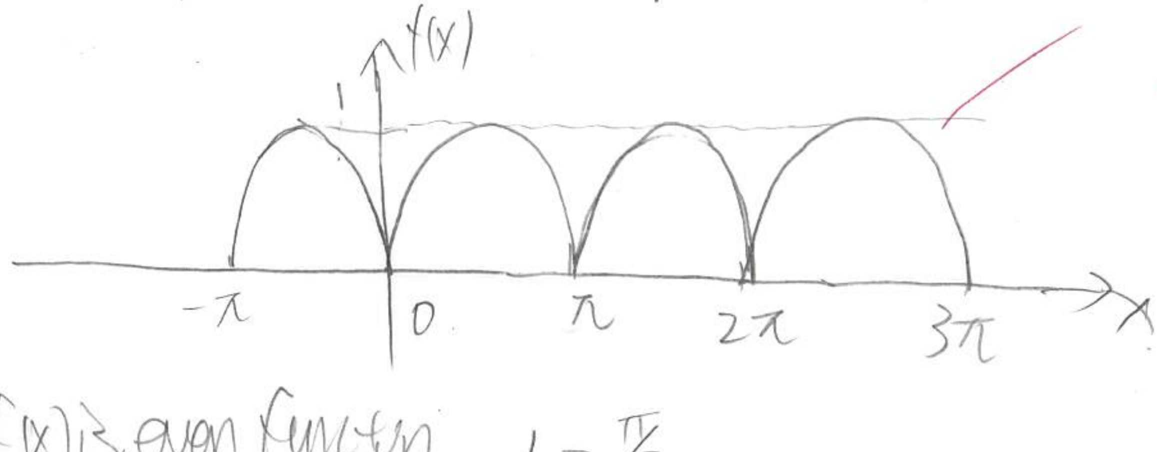
0 → not attempted

a). $T = \pi$

$f(x) = |\sin x|$, $f(-x) = |\sin(-x)| = |-\sin x| = \sin x = f(x)$

$f(x) = f(-x)$, $f(x) = |\sin x|$ is even function.

b)



6/6

c). f(x) is even function , $L = \pi/2$

$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin nx\, dx = 0$

for a_0, $f(x)$ is even

$$a_0 = \frac{1}{\pi/2}\int_{-\pi/2}^{\pi/2} f(x)\,dx = \frac{2}{\pi/2}\int_0^{\pi/2} \sin x\,dx$$

$$= \frac{4}{\pi}\int_0^{\pi/2} \sin x\,dx = \frac{4}{\pi}(-\cos x)\Big|_0^{\pi/2} = \frac{4}{\pi}$$

$$a_n = \frac{1}{\pi/2}\int_{-\pi/2}^{\pi/2} f(x)\cos\left(\frac{n\pi}{\pi/2}x\right)dx = \frac{2}{\pi/2}\int_0^{\pi/2}(\sin x)\,[\cos 2nx]\,dx$$

$$a_n = \frac{4}{\pi}\int_0^{\pi/2} \sin x \cos 2nx\,dx$$

$$\sin x \cos 2nx = \frac{1}{2}\left[\sin(x+2nx) - \sin(2nx-x)\right]$$
$$= \frac{1}{2}\left[\sin[(1+2n)x] - \sin[(2n-1)x]\right]$$

$$a_n = \frac{4}{\pi}\cdot\frac{1}{2}\int_0^{\pi/2}\left[\sin[(1+2n)x] - \sin[(2n-1)x]\right]dx$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\sin[(1+2n)x]\,dx - \frac{4}{\pi}\int_0^{\pi/2}\sin[(2n-1)x]\,dx$$

$$= \frac{2}{\pi}\left[\frac{-\cos(1+2n)x}{1+2n}\right]_0^{\pi/2} - \frac{4}{\pi}\left[\frac{-\cos(2n-1)x}{2n-1}\right]\Big|_0^{\pi/2}$$

$$= \frac{2}{\pi}\left[\frac{-\cos[(1+2n)\frac{\pi}{2}]+1}{1+2n}\right] - \frac{4}{\pi}\left[\frac{-\cos(2n-1)\frac{\pi}{2}+1}{2n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{(1-\cos(1+2n)\frac{\pi}{2})(2n-1) - (1-\cos(2n-1)\frac{\pi}{2})(2n+1)}{4n^2-1}\right]$$

BLANK PAGE

$$a_n = \frac{2}{\pi}\left[\frac{2n-1-(2n+1)}{4n^2-1}\right] = \frac{4}{\pi}\left[\frac{-1}{4n^2-1}\right]$$

$$L = \pi/2$$

$$\therefore f(x) \simeq \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cdot \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$$

$$f(x) \simeq \frac{\frac{4}{\pi}}{2} + \sum_{n=1}^{\infty}\frac{4}{\pi}\left[\frac{-1}{4n^2-1}\cos\frac{n\pi}{\pi/2}x\right]$$

$$\simeq \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{4n^2-1}\cos 2nx\right]$$

fourier series of $f(x) = |\sin x|$:

$$\frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{4n^2-1}\cos 2nx\right], (n=1,2,3\cdots)$$

12/12