Midterm 1, Math 170B, Winter 2018
Instructor: Georg Menz

Printed name: XIAOHE YANG

Signed name: xiaohe yang [illegible]

Student ID number: 504640737

Instructions:

- Read problems very carefully. If you have any questions raise your hand.
- You need to show all work and explain all reasoning in order to receive full credit.
- Clarity will also be considered in grading.
- You are allowed to use only the items necessary for writing. Notes, books, sheets, calculators, phones are **not** allowed, please put them away.
- If you need more paper please raise your hand (you can write on the back pages).
- Answers can be expressed in terms of binomial coefficients and factorials.

Question	Points	Score
1	6	6
2	4	4
3	4	3
4	4	4
5	4	4
Total:	22	21

1. (a) (2 points) Let X, Y be independent random variables with $\mathbb{E}[X] = 0$, $\mathbb{E}[Y] = 1$, $\text{var}(X) = 1$, $\text{var}(Y) = 2$. Find $\mathbb{E}[(X+Y)^2]$.

$$E[(X+Y)] = E[X] + E[Y] = 1$$

$$var[X+Y] = var(X) + var(Y) = 3$$

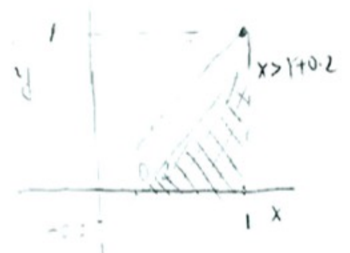
$$E[(X+Y)^2] - E[(X+Y)]^2 = var[X+Y]$$

$$E[(X+Y)^2] = 1^2 + 3 = \boxed{4}$$

✓

(b) (2 points) Let X, Y be independent random variables uniformly distributed on $[0, 1]$. Find $\mathbb{P}(X > Y + 0.2)$.

$P(X > Y+0.2)$ = area shaded on graph $= 0.8 \times 0.8 \times \frac{1}{2} = \boxed{0.32}$ ✓



$f_Y = f_X(h(y))\left|\frac{dh}{dy}\right|$

(c) (2 points) Let X be an exponentially distributed random variable with parameter $\lambda = 2$ i.e. $f_X(x) = 2\exp(-2x)$ for $x \geq 0$ and $f_X(x) = 0$ for $x < 0$. Find the PDF f_Y of the random variable $Y = X^{0.3}$.

$$f_X(x) = \begin{cases} 2e^{-2x} & x \geq 0 \\ 0 & \text{else} \end{cases}$$

$y = f(x) = x^{0.3}$

$x = h(y) = \sqrt[0.3]{y} = y^{\frac{10}{3}}$

$f(x) = X^{0.3}$ is monotonic decreasing

$h(y) = \sqrt[0.3]{y}$ is monotonic increasing

$$\frac{dh}{dy} = \frac{10}{3} y^{\frac{7}{3}}$$

$\because X = 2e^{-2x}$ in $x \geq 0$

$\therefore y = X^{0.3}$ in $y \geq 0$

$$f_Y(y) = f_X(h(y))\left|\frac{dh}{dy}\right| = \begin{cases} 2e^{-2y^{\frac{10}{3}}} \cdot \frac{10}{3} y^{\frac{7}{3}} & y \geq 0 \\ 0 & \text{else} \end{cases}$$

✓

6

Distance

2. (4 points) You are driving a boat in the LA bay. You drive the distance of 20 nautic miles at a constant speed which is uniformly distributed between 5 and 10 nautic miles per hour. What is the PDF of the duration of the trip?

Let X denote speed unif $[5,10]$

Y denote the duration of the trip.

$Y = \frac{20}{X}$ Let $f(x) = \frac{20}{x}$, $x = f^{-1}(x) = \frac{20}{Y}$

Let $g(y)$ denote $f^{-1}(x)$

$f_X(x) = \begin{cases} \frac{1}{5} & x \in [5,10] \\ 0 & \text{else} \end{cases}$

$\therefore \begin{cases} y = f(x) = \frac{20}{x} \\ x = g(y) = \frac{20}{y} \end{cases}$

CDF of Y: $F_Y(y) = P(Y \le y) = P(\frac{20}{X} \le y) = P(\frac{20}{y} \le X) = 1 - P(X < \frac{20}{y})$

$P(X < \frac{20}{y}) = \frac{\frac{20}{y}}{5} = \frac{4}{y}$ X unif $[5,10]$

when $y \in [2,4]$

$\therefore F_Y(y) = \begin{cases} 1 - \frac{4}{y} & y \in [2,4] \\ 0 & \text{else} \end{cases}$

✓

PDF of Y: $f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} \frac{4}{y^2} & y \in [2,4] \\ 0 & \text{else} \end{cases}$

4

$\frac{11}{6}+\frac{1}{4}$ $\frac{22+3}{12}$ $\frac{25}{12\times 4}$

3. (4 points) A four sided fair die is rolled and the outcome is recorded in the random variable D. Next, a coin with probability $1/D$ for heads is tossed 5 times and the number of times tail appeared is recorded in the random variable B. Find $\mathbb{E}[B]$.

D: 4-side die roll #

Q: $\frac{1}{D}$ 4-side die roll with $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$ P

B: # tail in rolling 5 times coin w/ $\frac{1}{D}$ head

$E[D] = \frac{1}{4}\times(1+2+3+4) = \frac{5}{2}$

$E[Q] = \frac{1}{4}\times\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right) = \frac{25}{48}$ ✓

$E[Q] = E[\frac{1}{D}]$

$E[B|D] = 5\times\frac{1}{D} = \frac{5}{D}$

$5(1-\frac{1}{D})$

$E[B] = E[E[B|D]] = E[\frac{5}{D}] = 5\,E[\frac{1}{D}] = 5\times\frac{25}{48} = \boxed{\frac{125}{48}}$

3

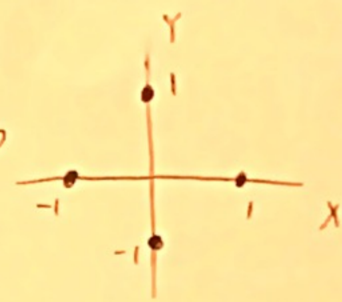
4. (4 points) Give an example of two random variables X and Y and proof that they are uncorrelated but not independent.

$$\rho_{X,Y} = \frac{cov(X,Y)}{\sqrt{var(X)}\sqrt{var(Y)}}$$

X, Y uncorrelated

$\rho_{X,Y} = 0$, $cov(X,Y) = 0$

Consider X, Y to be the x, y value of the 4 points shown in graph



$cov(X,Y) = E[X \cdot Y] - E[X]E[Y]$

$E[XY] = \frac{1}{4}\times1\times0 + \frac{1}{4}\times0\times1 + \frac{1}{4}\times(-1)\times0 + \frac{1}{4}\times0\times-1 = 0$

$E[X] = \frac{1}{4}\times1 + \frac{1}{4}\times(-1) + \frac{1}{4}\times0 + \frac{1}{4}\times0 = 0$

$E[Y] = \frac{1}{4}\times1 + \frac{1}{4}\times(-1) + \frac{1}{4}\times0 + \frac{1}{4}\times0 = 0$

$\therefore cov(X,Y) = E[XY] - E[X]E[Y] = 0$

then $\rho_{X,Y} = 0$, X, Y uncorrelated

However they are not independent.

Will show $var(X+Y) \neq var(X) + var(Y)$

$X+Y: \{1, 1, -1, -1\}$ $E[X+Y] = 0$

4p

$(X+Y)^2: \{1, 1, 1, 1\}$ $E[X+Y]^2 = 1$

Left $= var(X+Y) = 1 - 0 = 1$.

$var(X) = E[X^2] - E[X]^2 = 1 - 0 = 1$

$var(Y) = E[Y^2] - E[Y]^2 = 1 - 0 = 1$

Right $= var(X) + var(Y) = 1 + 1 = 2$.

$\therefore$ X, Y not independent

5. (a) (2 points) Show that the transform M_Y of the random variable $Y = 4X - 1$, where X a standard normal random variable, is given by

$$M_Y(s) = e^{8s^2 - s}.$$

$M_X(s) = e^{\frac{s^2}{2}}$ $\quad Y = 4X - 1$

$M_Y(s) = e^{-1 \cdot s} \cdot M_X(4s) = e^{-s} \cdot e^{\frac{(4s)^2}{2}} = e^{-s} \cdot e^{\frac{16s^2}{2}}$

$= e^{-s} \cdot e^{8s^2}$

$= e^{8s^2 - s}$

2p

(b) (2 points) The transform M_X of a random variable is given by

$$M_X(s) = e^{8s^2}\left(\frac{1}{4} + \frac{1}{4}e^{2s} + \frac{1}{2}e^{4s}\right).$$

$e^{\frac{s^2}{2}}$

How is the random variable X distributed?

$M_{X_1}(s) = e^{8s^2}$ $\quad M_{X_2}(s) = \frac{1}{4}e^{0s} + \frac{1}{4}e^{2s} + \frac{1}{2}e^{4s}$

$M_X(s) = M_{X_1}(s) \times M_{X_2}(s) \implies X = X_1 + X_2.$

X_1: $\quad M_{X_0}(s) = e^{\frac{s^2}{2}}$ $\quad X_0\ N(0,1)$ $\quad M_{X_1}(s) = e^{\frac{(4s)^2}{2}}$ $\quad X_1 \sim 4X_0$

X_2: discrete $\quad p(X_2) = \begin{cases} \frac{1}{4} & x = 0 \\ \frac{1}{4} & x = 2 \\ \frac{1}{2} & x = 4 \end{cases}$

2p

$X = 4X_0 + Y_2$ where X_0 is $N(0,1)$

Y_2 is discrete with PMF

$p(x) = \begin{cases} \frac{1}{4} & x = 0 \\ \frac{1}{4} & x = 2 \\ \frac{1}{2} & x = 4 \end{cases}$