Math 170A: Probability Theory, A. Iseli   17 March 2020

Final Exam (Duration: 3 hours)

Read this page carefully before taking the exam! (this is before the 3 hours)

Before the exam:

- Recall the policies for this exam (see last page).
- make sure you have a calm workspace with internet access and enough paper

Taking the exam:

- Carry out all steps in proofs and computations and justify all answers. This includes pointing out where you use the given assumptions.
- please start a new page for each problem (but not necessarily for each part a,b,c)
- clearly indicate which problem you are working on on each of your pages (no need to copy the statement of the problem though)
- your solutions should be based on theory and techniques taught in this course.
- the duration is 3 hours. Set yourself a timer once you start reading the problems.

After the exam:

- write your name and UID legibly on the first page.
- copy (i.e. write by hand) the below Student Conduct Code (SCC) statement on the first or last page
- add today's date, your name, and your signature below the SSC statement
- number your pages on the upper right corner
- scan/photograph your solutions and convert them into one legible, reasonable size pdf file
- upload your pdf file to Gradescope or CCLE (either one is fine with me)
- in case something does not work, please do not panic, we will find a solution. Just email me as quickly as possible.
- **Do not talk about the exam to anybody before the 24h window is over.**

Student conduct code statement:

I assert, on my honor, that I have not received assistance of any kind from any other person while working on the final, that I have not worked on the final for more than 3 hours (in one piece), and that I have not used any non-permitted materials or technologies during the period of this evaluation. I am aware of UCLA's student conduct code and of the policies communicated by the instructor.

[Full name, signature, today's date]

Question	Points	Score
1	8	
2	10	
3	10	
4	8	
5	10	
6	6	
7	12	
8	21	
Total:	85	

1. (a) (4 points) Let $\Omega = \{1,2,3,4\}$ and $\tilde{\mathscr{F}} = \{\{1,2\},\{3,4\},\{1,2,3,4\},\{1\},\{2\}\}$ a family of subsets. How many and which sets do you need to add to $\tilde{\mathscr{F}}$ to make it an event space? (State which sets you need to add and why you need to add them.)

   (b) (4 points) We denote the event space from part (a) by $\mathscr{F}$. Define two different probability measures P and $\tilde{P}$ on $(\Omega, \mathscr{F})$.

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2. The Kerckhoff cafe on UCLA campus offers k new flavors of iced lattes. Everyday, one of your fellow students offers you an iced latte of a random flavor. You cannot choose which flavor you receive, and every day you receive each flavor with the same probability, independently. Let X be the number of days until you have tasted all flavors of iced latte.

   (a) (6 points) What is the expectation and variance of X? Properly define all objects that you are introducing. (Hint: Coupon collecting problem.)

   (b) (4 points) Find a good lower bound for the probability of X lying between $\mathbb{E}(X) - 1$ and $\mathbb{E}(X) + 1$?

   (c) (Bonus) Explain in two to three sentences what part (b) means for your latte experience. (This answer has not a unique perfect solution. I just want to hear a bit how you connect your computational result to the example. A sentence or two is enough.)

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3. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} \frac{3}{4}x^2 & \text{if } 0 < x < y < 2 \\ 0 & \text{else} \end{cases}$$

   (a) (2 points) Show that f is a joint density function.

   (b) (2 points) Find the joint distribution function $F_{X,Y}$ for X and Y.

   (c) (4 points) Find the marginal densities for X and Y.

   (d) (2 points) Find the density of $X + Y$.

4. (8 points) In the course of a year, X baby turtles are born. Assume that X has Poisson distribution with parameter $\lambda = 2$. Each baby turtle will survive the first year with probability $\frac{1}{2}$. Let Y be the number of baby turtles alive after the first year. Find $P(Y = n)$ for each $n \in \mathbb{N}$ and find $\mathbb{E}(Y)$. (Hint: condition on X.)

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5. Let X be continuously uniformly distributed on $[a, b]$ for $0 < a < b$ are constants in $\mathbb{R}$.
   (a) (5 points) Find the distribution function of $Y = \frac{1}{X}$. Is Y a continuous random variable? If so, what is its density function?
   (b) (5 points) Given an example of a continuous random variable X and a function $g : \mathbb{R} \to \mathbb{R}$, so that $Y = g(X)$ is a random variable that is not continuous.

   (Prove that for your choice of X and g, $Y = g(X)$ it is a random variable and check that it is not continuous.)

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6. (6 points) Let X, Y be random variables. Suppose that for all $a, b, c, d \in \mathbb{R}$ the following holds: $P(a < X \leq b, c \leq Y \leq d) = P(a < X \leq b)P(c < Y \leq d)$.
   Prove that $P(X \leq r, Y \leq fs = P(X \leq r)P(Y \leq s)$ for all $r, s \in \mathbb{R}$.

   (Hint: continuity of probability measures)

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7. Consider the following joint density function

$$f(x, y) = \begin{cases} \frac{2}{4+\pi}(y + \sin(x)) & \text{if } 0 < x < \pi \text{ and } 0 < y < 1 \\ 0 & \text{else} \end{cases}$$

   You do not need to prove that f is a joint density function. You may take that for granted.
   (a) (4 points) Find the conditional density of Y given $X = x$ for all values of $x \in \mathbb{R}$ where it is defined, and find the conditional density of X given $Y = y$ for all values of $y \in \mathbb{R}$ where it is defined.
   (b) (4 points) Compute expectation and variance of both X and Y?
   (c) (4 points) Are X and Y independent? Are they uncorrelated?

8. (21 points) **True or false.**

For each of the statements below, decide whether it is true or false. Moreover, give a explanation or proof, or cite a relevant definition or theorem, or mention an example or counterexample that justifies your claim.

For parts (a) - (d): Let X and Y be two continuous random variables for which expectation, variance, and covariance of $X, Y, X+Y$, and XY exist.

(a) If $\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$ then $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$.

(b) If X and Y are uncorrelated, then they are independent.

(c) If X and Y are independent, then they are uncorrelated.

(d) Assume $\mathrm{Var}(X) \neq 0$ and $\mathrm{Var}(Y) \neq 0$. Then: $\mathrm{Cov}(X, Y) = 0$ if and only if the correlation coefficient $\rho(X, Y) = 0$.

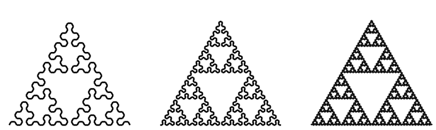
(e) There exists a random variable X that is both discrete and continuous.

(f) There exists a random variable Y that is neither discrete nor continuous.

(g) There exists a function that is the density function of a random variable, as well as the distribution function of a (possibly different) random variable.

I enjoyed working with all of you this quarter. Thank you.

Have a good spring break and stay safe!



Rules and policies for the final exam (170A, Section 003, A.Iseli)

- The final is an **open book exam**: you are allowed to use any source available (books, lecture notes, homework, google, wikipedia, wolframalpha.com,...).
- **Collaboration is prohibited**: in particular, do not ask other people for help for hints or to solve the problems for you, don't share your solutions with others, do not post or discuss problems related to the final on any sort of (online) platform.
- Your solutions should be based on **theory and techniques taught in this course**: applying theorems that were not covered in this course (and that you cannot easily prove using the techniques and theorems from this course) will result into little to zero credits for the respective problem.
- The **duration is 3h**: you have three hours to work on the final. These three hours must lie between 8am on Tuesday and 8am on Wednesday. We appeal to your sense of fairness and honesty to not use more time than the foreseen three hours.
- You are **encouraged to use the original time frame** of the exam, which is 8-11am on Tuesday. Whoever does not wish to participate in this, please write me an email until Monday, March 16, 6pm. If you must change something about your exam time last minute, please inform me immediately by email. I can accommodate emergency requests as long as they lie within the 24 hour window.
- **Referencing is strongly encouraged**: e.g. if in a problem you are asked to come up with an example for something and you happen to find one on (say) math stack exchange, I prefer if you let me know about the source. This won't result in a loss of points for you, and it might actually prevent further questions from my side during the grading process (see next bullet).
- The instructor has the **right to ask for explanations** of your solutions at any point of the grading process. In particular, for people who have a large gap between downloading ttime and uploading time on Gradescope, I will use my right to ask questions as I know that having to google too many of the problems takes a lot of time.
- Statement of **Students Conduct Code (SCC)**: every exam at UCLA requires you to act according to the SSC:
  `https://www.deanofstudents.ucla.edu/Individual-Student-Code`.
  The SCC is an agreement that you signed when you became a UCLA student. Amongst other things, it states that you follow rules and policies set by the department and the instructor. In particular, this includes following the instructions on the first page of the exam file.