Midterm Examination I Mathematics 170A [redacted]

Name yinghao Wang Oct. 29, 2018

Student no: 204983840

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1 Consider the 36-outcome space of the results of throwing two dice.

Let X = the number the first die shows and Y= the number the second die shows.

(A) Are X and Y independent random variables? Explain in detail why or why not.

(B) If Z= X+Y, are X and Z independent random variables? Explain why or why not carefully(by actually computing some probabilities).

A) X and Y are independent random variables.

$E(x) = 3.5,\ E(y) = 3.5$

Does $E(X \cdot Y) = E(x) \cdot E(Y)$?

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	8	10	12
3	3	6	9	12	15	18
4	4	8	12	16	20	24
5	5	10	15	20	25	30
6	6	12	18	24	30	36

$$E(X \cdot Y) = \frac{36+2(30)+2(24)+25+2(18)+2(20)+2(12)+2(15)+16+2(6)+2(10)+2(12)+2(5)+2(8)+9+2(4)+2(6)+2(3)+4+4+1}{36}$$

$12.25 = \frac{441}{36}$

$36\overline{)441}$ = 12.5; 36, 81, 72, 9

$E(x) \cdot E(Y) = 12.25,\ E(X \cdot Y) = 12.25$, showing X, Y are independent

Also, $P(X=a \cap Y=b) = P(X=a) \cdot P(Y=b)$ Why? -2

B) ON BACK ⇒

Z	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$$\begin{array}{r} {}^{3}\,3.5 \\ 7 \\ \hline 24.5 \end{array}$$

X and Z are not independent random variables. The value of Z depends on the value of X. For instance, consider $P(Z=12 \cap X=4)$. If Z and X were independent, that probability should equal $P(Z=12) \cdot P(X=4)$. But, clearly $P(Z=12 \cap X=4) = 0$ because Z can't be 12 if X is 4 (since Y only goes up to 6). But, $P(Z=12) \cdot P(X=4) = \frac{1}{36} \cdot \frac{1}{6}$, which is not 0.

Therefore, X and Z aren't independent random variables.

2 (A) Explain what the Bernoulli random variable with probability p ,0< p< 1, is and calculate its expected value and variance.
(B) State the theorems on the expected value of the sum of random variables and the variance of the sum of independent random variables.
(C) Explain how parts A and B together enable one to calculate easily the expected value of the binomial distribution of n trials with probability p for success in each trial(i.e. the distribution ~~is~~ in problem 5).

A) Bernoulli random variable is 1 with probability p. and 0 with probability $1-p$. It's expected value is $1 \cdot p + 0(1-p) = p$, and it's variance is $E(x^2) - (E(x))^2 = p - p^2 = p(1-p)$. ($E(x^2) = E(x)$ since x^2 can only be 0, 1)

B) $E(X_1 + X_2) = E(X_1) + E(X_2)$

$var(X_1 + X_2) = Var(X_1) + var(X_2)$

c) The binomial distribution is the sum of n independent Bernoulli random variables.

So, the expected value is $E(X_1 + X_2 + \dots + X_n) = E(X_1) + E(X_2) + \dots + E(X_n) = np$.

The variance is $var(X_1 + X_2 + \dots + X_n) = var(X_1) + var(X_2) + \dots + var(X_n) = np(1-p)$

3 Five cards are dealt from a standard 52 card deck.
(A) What is the probability that four of them are of a single suit and the fifth is a different suit? Explain your reasoning carefully –how you got the answer. (The answer can be written in terms of binomial coefficients and so on—a numerical calculation is not necessary)
(B) What is the probability that four are of the same suit, the fifth not, and the four of the same suit are all between 2 and 10 in rank.

A) $\boxed{\dfrac{\binom{4}{1}\binom{13}{4}\binom{3}{1}\binom{13}{1}}{\binom{52}{5}}}$

$\binom{4}{1}$ = # ways to choose 1st suit

$\binom{13}{4}$ = # ways to choose 4 cards of the same suit

$\binom{3}{1}$ = # ways to choose 2nd suit

$\binom{13}{1}$ = # ways to pick last card (any card out of a suit)

$\binom{52}{5}$ = total # of 5 card hands

B) $\boxed{\dfrac{\binom{4}{1}\binom{9}{4}\binom{3}{1}\binom{13}{1}}{\binom{52}{5}}}$

$\binom{13}{1}$ = # ways to choose a card out of a suit

$\binom{52}{5}$ = # 5 card hands

$\binom{4}{1}$ = # ways to choose 1st suit

$\binom{9}{4}$ = # ways to choose 4 cards between 2 and 10 (inclusive)

$\binom{3}{1}$ = # ways to choose 2nd suit

1 3.5

$24\overline{)324}$ 24 84 72 120 $\frac{1}{24}$, $\frac{1}{14.5}$ 14.5 324 324

$$\frac{\frac{1}{24}}{\frac{1}{24}+\frac{1}{324}}$$

36 6 216 3 648

4 A die is weighted so that it comes up 6 half the time(probability 1 /2) and the other values 1.2.3.4,5 each come up with probability 1/10. Two other fair dice and the weighted die are presented and one is chosen at random. The randomly chosen one is thrown three times and comes up a 6 each of the three times. What is the probability that the randomly chosen die is actually the weighted one?

Want to find P(weighted | three 6's)

$$P(\text{weighted} \mid \text{three 6's}) = \frac{P(\text{three 6's} \mid \text{weighted}) \cdot P(\text{weighted})}{P(\text{three 6's})}$$

$$\boxed{\frac{\left(\frac{1}{2}\right)^3 \cdot \frac{1}{3}}{\frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{3}\left(\frac{1}{6}\right)^3 + \frac{1}{3}\left(\frac{1}{6}\right)^3}}$$

P(three 6's) = P(weighted die and 3 6's) + 2·P(fair die and 3 6's)

5 (A) Define the binomial distribution with probability p, n trials ,$0<p<1$. and show that it really is a probability distribution(i.e., that the sum of all the probabilities is 1)
(B) When p= 1/2 , the binomial distribution is a representation of the probability of k heads in n coin tosses. If a coin is tossed 10 times, what is the probability that three or fewer heads occur in the 10 tosses? What is the probability that exactly 5 heads occur?
(C) What is the probability that there are more heads than tails in the ten tosses?

A) The binomial distribution is $\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$

By the binomial theorem, this is $(p+1-p)^n =$

$1^n = 1$

B) 3 or fewer heads

$\sum_{k=0}^{3} \binom{10}{k} (\tfrac{1}{2})^k (\tfrac{1}{2})^{n-k} = \binom{10}{0}(\tfrac{1}{2})^0(\tfrac{1}{2})^{10} + \binom{10}{1}(\tfrac{1}{2})(\tfrac{1}{2})^9 + \binom{10}{2}(\tfrac{1}{2})^2(\tfrac{1}{2})^8 +$
$\binom{10}{3}(\tfrac{1}{2})^3(\tfrac{1}{2})^7$

Exactly 5 heads

$\binom{10}{5}(\tfrac{1}{2})^5(\tfrac{1}{2})^5$

c) consider $k=6$ to $k=10$

$\sum_{k=6}^{10} \binom{10}{k}(\tfrac{1}{2})^k(\tfrac{1}{2})^{n-k} = \binom{10}{6}(\tfrac{1}{2})^6(\tfrac{1}{2})^4 + \binom{10}{7}(\tfrac{1}{2})^7(\tfrac{1}{2})^3 + \binom{10}{8}(\tfrac{1}{2})^8(\tfrac{1}{2})^2 +$
$\binom{10}{9}(\tfrac{1}{2})^9(\tfrac{1}{2}) + \binom{10}{10}(\tfrac{1}{2})^{10}(\tfrac{1}{2})^0$