Math 132 Midterm
May 6, 2011

82 +3

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Instructions: Show all of your work, and clearly indicate your answers. Use the backs of pages as scratch paper. No books, other paper, or calculators are allowed.

1. (20 points) Express the following complex numbers in either polar form or standard form.
   (a). $i^{\frac{1}{5}}$.
   (b). $\text{Log}(1+i)$.
   (c). i^i.
   (d). 2^i.

a. $e^{\frac{1}{5}\log i} = e^{\frac{1}{5}(\cancel{\log|i|} + i\frac{\pi}{2} + i2\pi k)} = \boxed{e^{i\left(\frac{\frac{\pi}{2}+2\pi k}{5}\right)}}$

b. $\text{Log}(1+i) = \log|1+i| + i\frac{\pi}{4} = \boxed{\log\sqrt{2} + \frac{i\pi}{4}}$

c. $i^i = e^{i\log i} = e^{i(\cancel{\log|i|} + i\frac{\pi}{2} + i2\pi k)} = \boxed{e^{-\frac{\pi}{2} - 2\pi k}}$

d. $2^i = e^{i\log 2} = e^{i(\log 2 + i2\pi k)} = \boxed{e^{i(\log 2)}e^{-2\pi k}}$

log 2 is an angle

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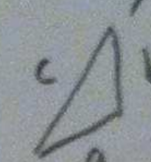
2. (20 points) Prove the triangle inequality

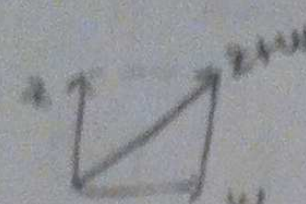
$$|z+w| \le |z| + |w|$$

for any two complex numbers z and w. When the equality holds?

We know from geometry



lengths $a + b \geq c$ $\forall a, b$ in a triangle



z and w when added together make z+w

use complex numbers to argue

So the analog to above is

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$|z| \sim a$

$|w| \sim b$

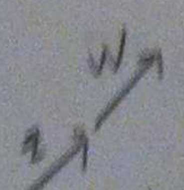
$|z+w| \sim c$

so

$|z| + |w| \geq |z+w|$

for any two complex numbers z and w

The case of equality are when $\underline{\operatorname{Arg} z = \operatorname{Arg} w}$
or the two ~~numbers have the~~ same angle



$|z| + |w| = |z+w|$

3. (20 points) Show that if both f and $\bar{f}$ are analytic, then f is a constant.

$f = x + iy$

$\bar{f} = x - iy$

$g = \dfrac{f + \bar{f}}{2} = \dfrac{x \cancel{+iy} + x \cancel{-iy}}{2} = x$

$h = \dfrac{f - \bar{f}}{2i} = \dfrac{\cancel{x} + iy \cancel{-x} + iy}{2i} = y$

If f and $\bar{f}$ are analytic, then g and h are analytic because the sum of analytic functions is analytic

But g and h are also real valued (because the imaginary parts are gotten rid in the definition)

Any function that is analytic and real valued is constant

$U_x = V_y$ V being the imaginary part of g and h which $= 0$

$U_y = -V_x$

So $g_x = 0$

$g_y = 0$

$h_x = 0$

$h_y = 0$

So $g = c$ c, d are constants

$h = d$

Therefore $f = c + id$ which is constant

+20

4. (20 points) Show that $xy + 3x^2y - y^3$ is harmonic and find its harmonic conjugate.

$U = xy + 3x^2y - y^3$

$U_x = y + 6xy$ $\qquad\qquad U_y = x + 3x^2 - 3y^2$

$U_y = -V_x$

$V_x = -x - 3x^2 + 3y^2$ $\quad$ integrate wrt x

$V = -\frac{x^2}{2} - x^3 + 3y^2x + g(y)$ $\quad$ differentiat wrt y

$V_y = 6yx + g'(y) \quad = U_x = y + 6xy$

from CR eq

$g'(y) = y$

$g(y) = \frac{y^2}{2} + C$

$$\boxed{V(x,y) = -\frac{x^2}{2} - x^3 + 3xy^2 + \frac{y^2}{2} + C}$$

✓ V is U's harmo conjugate

$U_{xx} = 6y \qquad\qquad U_{yy} = -6y$

$U_{xx} + U_{yy} = 6y - 6y = 0 \quad$ ✓ $\therefore$ U is harmonic

+20

5. (20 points) Given any z_0 in the complex plane. Evaluate the integral

$$\oint_{|z-z_0|=2} \frac{e^z}{(z-z_0)^m} dz$$

$$\oint \frac{f(\omega)}{(\omega - z)^{m+1}} d\omega =$$

where m is an integer.

Cauchy Integral Formula

$$f^{(m')}(z) = \frac{m'!}{2\pi i} \oint_{\partial D} \frac{f(w)}{(w-z)^{m'+1}} dw$$ in common form

use m' to not get confused with m

Cauchy Integral Formula to match

$$\cancel{\oint_{|z-z_0|=2} \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i f^{n}(z_0)}{n!}}$$

$m = m'+1$ $\qquad f(z) = e^z$

$m' = m-1$

$$\oint_{\partial D} \frac{e^z}{(z-z_0)^m} dz = \frac{f^{(m-1)}(z_0)\, 2\pi i}{(m-1)!}$$

$z_0 \in |z-z_0| = 2$ so the integral always has a value

$(m-1)^{th}$ derivative

$$\frac{f^{(m-1)}(z_0)\, 2\pi i}{(m-1)!}$$

$$\frac{\partial^{(m)} e^z}{\partial z^{(m-1)}} = e^z$$

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$$\boxed{\frac{2\pi i}{(m-1)!} e^{z_0}}$$

$$\frac{m \geq 0}{m < 0}$$