1. (10 points) Prove the following identity using methods of complex analysis:

$$\cos(3\theta) = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta)$$

There were two valid interpretations:
① θ is real ② θ is a general complex number

Interpretation 1:

$$e^{3i\theta} = (e^{i\theta})^3 \quad \& \quad e^{i\theta} = \cos\theta + i\sin\theta \Rightarrow$$

$$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$$
$$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta \quad (*)$$

$\Rightarrow \cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ since the real & imaginary parts must be equal in $(*)$

Interpretation 2: $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ let $q := e^{i\theta}$ $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

Then $\cos^3\theta = \left(\dfrac{q+q^{-1}}{2}\right)^3 = \dfrac{1}{8}(q^3 + 3q + 3q^{-1} + q^{-3})$

$$3\cos\theta\sin^2\theta = 3\left(\frac{q+q^{-1}}{2}\right)\left(\frac{q-q^{-1}}{2i}\right)^2 = -\frac{3}{8}(q+q^{-1})(q-q^{-1})(q-q^{-1})$$
$$= -\frac{3}{8}(q^2 - q^{-2})(q-q^{-1}) = \frac{-3}{8}(q^3 - q^{-1} - q + q^{-3})$$

$$\Rightarrow \cos^3\theta - 3\cos\theta\sin^2\theta = \frac{1}{8}(4q^3 + 4q^{-3})$$
$$= \frac{1}{2}(e^{3i\theta} + e^{-3i\theta})$$
$$= \cos 3\theta$$

2. (10 points) This question has parts (a) and (b). As usual, let $z = x + iy$.

(a) (5 points) Find an entire function whose real part is $x^2 + 2xy - y^2$.

let $u = x^2 + 2xy - y^2$ $\quad u_x = v_y \Rightarrow 2x + 2y = v_y$

$\Rightarrow v = \int (2x+2y)\,dy = 2xy + y^2 + C(x)$. Now $u_y = -v_x$

$\Rightarrow \frac{d}{dx}\left(2xy + y^2 + C(x)\right) = 2y + C'(x)$

$= -\frac{d}{dy}\left(x^2 + 2xy - y^2\right) = 2y - 2x$

$\Rightarrow C'(x) = -2x \Rightarrow C(x) = -x^2 + k \quad k \in \mathbb{C}$.

Then $v = y^2 + 2xy - x^2 + k$. A possible analytic function is therefore

$$f(x+iy) = x^2 + 2xy - y^2 + i\left(y^2 + 2xy - x^2 + k\right)$$

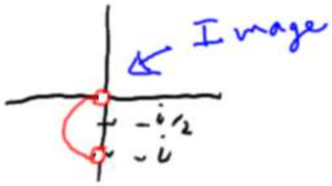
(b) (5 points) Explain why there is no entire function whose real part is $2xy - y^2$.

$$\Delta(2xy - y^2) = \nabla^2(2xy - y^2) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(2x - y^2)$$

$$= (0 - 2) = -2 \Rightarrow 2xy - y^2 \text{ is}$$

not harmonic. Therefore it cannot be the real part of any analytic function on any domain in $\mathbb{C}$, and consequently cannot be entire.

3. (10 points) This problem has parts (a)-(c). Let f be the Möbius transform



$$f(z) = \frac{1}{z+i}.$$

(a) (3 points) What is the image of the negative real line, $\{z \in \mathbb{C} \mid \mathrm{Re}(z) < 0\}$, under this transformation? $\&\ \mathrm{Im}(z) = 0$

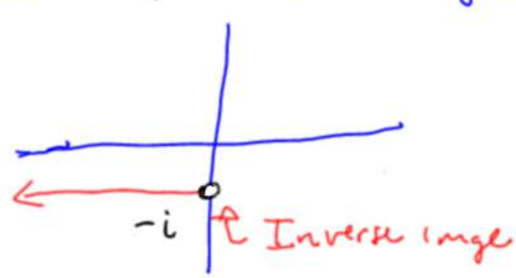
If you add i to z the negative real line shifts up one unit to become $\{z \in \mathbb{C} \mid z = x + i,\ x < 0\}$ which is part of the line $\{z \mid \mathrm{Im}\, z = 1\}$. Then perform an inversion on $\{z \mid \mathrm{Im}\, z = 1\}$ which goes to the circle through the origin with furthest point $1/i = -i$ (i was the closest point to 0).
This is the circle $|z + i/2| = 1/2$. However our image corresponds to the negative half real axis, so our image should be half of the circle. the neg. half line goes from 0 to $-\infty$ and thus there are two possibilities $\{z \mid |z + i/2| = \frac{1}{2},\ \mathrm{Re}\, z < 0\}$ and $\{z \mid |z + i/2| = \frac{1}{2}\ \mathrm{Re}\, z > 0\}$. We see that $f(-1) = \frac{1}{-1-i} = -\frac{1}{2} - \frac{1}{2}i$ which lies in the first set, so
the image is the left semicircle with 0 & $-i$ excluded.

(b) (4 points) What is the **inverse** image of the negative real line, $\{z \in \mathbb{C} \mid \mathrm{Re}(z) < 0\}$, $=: \mathbb{R}_-$ under this transformation? $\&\ \mathrm{Im}(z) = 0$

$$\omega = \frac{1}{z+i} \Rightarrow \omega(z+i) = 1$$
$$\omega z + \omega i = 1$$
$$\omega z = 1 - \omega i$$
$$z = \frac{1 - \omega i}{\omega} \Rightarrow f^{-1}(z) = \frac{1}{z} - i$$

$\frac{1}{z} \in \mathbb{R}_-$ if $z \in \mathbb{R}_-$. $\Rightarrow$ The image of $\mathbb{R}^{-1}$ is just

$\{x - i1 \mid x \in \mathbb{R}\ \&\ x < 0\}$



(c) (3 points) Let $\mathrm{Log}(z)$ be the standard branch of the logarithm. Where is the function $\mathrm{Log}(\frac{1}{z+i})$ analytic?

$\mathrm{Log}(\frac{1}{z+i})$ is analytic when the image of $\frac{1}{z+i}$ avoids the principle branch of $\mathrm{Log}\, z$, the negative real axis. This is precisely the inverse image of the negative real axis. By part (b) this is $\{z \in \mathbb{C} \mid \substack{\mathrm{Im}\, z = -1 \\ \mathrm{Re}\, z < 0}\}$.

Thus $\mathrm{Log}(\frac{1}{z+i})$ is analytic on

$$\mathbb{C} \setminus \{z \in \mathbb{C} \mid \mathrm{Im}\, z = -1 \;\&\; \mathrm{Re}\, z < 0\}$$

4. (10 points) This question has parts (a)-(c).

(a) (3 points) Show that the function $f(x+iy) = 2xy$ is not complex differentiable at any point other than the origin.

$f(x+iy) = u(x+iy) + i\,v(x+iy)$ with $u = 2xy$, $v = 0$

If f were complex differentiable at $x+iy$, then

$u_x = v_y$ & $u_y = -v_x \Rightarrow 2y = 0$ & $2x = 0$

$\Rightarrow x+iy = 0 \Rightarrow$ the only possible point where f is complex differentiable is the origin.

(b) (4 points) Show, using the definition of the derivative, that the function $g(x+iy) = x^2+y^2$ is complex differentiable at the origin. (Hint: the formula $x^2+y^2 = (x+iy)(x-iy)$ may be useful.)

In order to be cplx. diff. at 0, we must have that the limit $\lim_{\Delta z \to 0} \frac{g(0+\Delta z) - g(0)}{\Delta z}$ exists

this is $\lim_{\Delta x + i\Delta y \to 0} \frac{\Delta x^2 + \Delta y^2}{\Delta x + i\Delta y} = \lim_{\Delta x + i\Delta y \to 0} \frac{(\cancel{\Delta x + i\Delta y})(\Delta x - i\Delta y)}{\cancel{\Delta x + i\Delta y}}$

$= \lim_{\Delta x + i\Delta y \to 0} \Delta x - i\Delta y = 0$.

Thus g is cplx. diff at 0.

(c) (3 points) Is the function g of part (b) complex analytic at the origin? Justify your answer.

As in part (a) we see $u = x^2+y^2$ & $v \equiv 0$

$$\Rightarrow \left.\begin{array}{l} u_x = v_y \Rightarrow 2x = 0 \\ u_y = -v_x \Rightarrow 2y = 0 \end{array}\right\} \Rightarrow x = 0 \;\&\; y = 0 \text{ if}$$

the function g is to be complex analytic.
But a function is cplx. analytic at 0 only if it is cplx. diff. on some open ball around 0.
Since the Cauchy Riemann equations shows that g is not complex diff. anywhere away from 0.
Thus, it cannot be complex analytic at zero.

5. (10 points) This question has parts (a) and (b). Let $f(z) = e^{\frac{1}{3}\mathrm{Log}(z)}$ be the principal branch of the cubed root function $z^{\frac{1}{3}}$.

(a) (5 points) find $f(-1)$

$$f(-1) = e^{\frac{1}{3}(\ln|-1| + i\,\mathrm{Arg}(-1))}$$
$$= e^{1/3(0 + i\pi)}$$
$$= e^{i\pi/3}$$

(b) (5 points) Give an example of a branch, $g(z)$, of the cubed root function where $g(1) = e^{2\pi i/3}$.

$$g(1) = e^{\frac{1}{3}\log(1)}$$
$$= e^{1/3(\ln 1 + i\arg(1))}$$
$$= e^{1/3(0 + i(0 + 2\pi k))} \quad k \in \mathbb{Z}$$
$$= e^{2\pi k/3\, i}$$

Thus we must select a branch of log where $2\pi \in (\alpha, \alpha + 2\pi]$. If we let $\alpha \in [0, 2\pi)$ we will satisfy the condition.

Let the branch be $e^{\frac{1}{3}\mathrm{Log}_{(\alpha,\alpha+2\pi]} z}$ defined away from the set $\{\rho e^{i\alpha} \mid \rho \geq 0 \;\&\; \alpha \in [0, 2\pi]\}$

Math 132 | Midterm | Winter 2014

First and Last Name: ______________________ Initial of Last Name: ________

Student ID: __________________

Instructions:

- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Circle or otherwise indicate your final answers. If you use the back of a page please indicate that you have work on the reverse side.
- You may not use books, notes, calculators, mobile phones, or any outside help during this exam. You may not collaborate with other students in any way during this exam.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total	50	