Practice Final Exam Solutions

Question 1:

(a) $f(z) = 2z/(z+1)$; $f(0) = 2 * 0/(0+1) = 0$; $f(1) = 2 * 1/(1+1) = 1$; $f(-1) = 2(-1)/(-1+1) =$ infinity

(b) $\{z: Re(z) < 1\}$ (The open half plane to the left of 1)

(c) $(-1+2i)/5$

(d) Simple pole at -1; removable singularity at infinity (Or: if f(infinity) is defined to be 2, then there is no singularity at infinity).

Question 2:

(a) Differentiable at (0,0); nowhere analytic

(b) No such function exists. (u is not harmonic).

(c) Errata for this question: It should be v, not u. Yes, a function does exist: $h(z) = (-x^2-2xy+y^2) + i(x^2 - 2xy - y^2) = (i-1)z^2$ will work.

Question 3:

(a) $1/2 \ln(2) + 3\pi i/4 + 2k\pi i$ (k is any integer)

(b) Everywhere except when $Im(z)=1$ and $Re(z) \geq 0$ (i.e. the ray to the right of i)

(c) Many examples work, e.g. $L_{-2\pi} (i - z)$. To see why this works, we compute

$L_{-2\pi} (i - 1) = \ln |i-1| + i \arg_{(-2\pi,0]} (i-1) = \ln \sqrt{2} + i (-5 \pi/4)$

Question 4:

If $|f(z)| > M$ for all z, then $|1/f(z)| < 1/M$ for all z. Since f is never zero, and f is entire: $1/f(z)$ is entire $1/f(z)$ is entire and bounded, hence by Louville's theorem it is constant. Hence f(z) is constant.

Question 5:

(a) If $0 < |z| < 1$:

$1/z^3 - 1/z^2 + 1/z - 1 + z - z^2 + \ldots$

If $|z| > 1$:

$1/z^4 - 1/z^5 + 1/z^6 - \ldots$

(b) Triple pole at 0; simple pole at -1; Res(0) = 1; Res(-1) = -1

Question 6:

(a) Simple zero at $-\pi/2$, removable singularity at $\pi/2$ (residue = 0)

(b) Essential singularity at 0 (residue = 1/2)

(c) Removable singularity at 0 (residue = 0); once singularity is removed, we have a simple zero

(d) Triple pole at 0 (residue = 2).

Problem 7:

(a) Possible parameterizations include

```
z(t) = 2 exp(it) [ 0 <= t <= 4 pi ]

z(t) = 2 exp(2 pi it) [ 0 <= t <= 2 ]

z(t) = 2 exp(4 pi it) [ 0 <= t <= 1 ]
```

(b) (Solution A) |z| is not differentiable, so it does not have an anti-derivative, and so none of the fancy theorems apply; we have to compute this by hand. Using the first parameterization z = 2 exp(it), dz = 2 i exp(it) dt in (a) we can write

```
 \int_Gamma |z| dz = int_0^{4 pi} |2 exp(it)| 2 i exp(it) dt
= int_0^{4 pi} 2 2 i exp(it) dt = 4 exp(it)|^{4pi}_0 = 0.
```

(Solution B) On Gamma |z| = 2, so we have

```
\int_Gamma |z| dz = \int_Gamma 2 dz.
```

But this is zero, because 2 is analytic on Gamma and inside Gamma (or alternatively, because 2 has an antiderivative on Gamma).

(c) We can write the integral as f(z) / (z-0) where f(z) = (z + exp(z))/(z-4). f is analytic on and inside Gamma, and 0 is inside Gamma. However Gamma is not simple (it wraps around the circle twice), so the CIF does not apply directly. Because Gamma is made up of two simple closed positively oriented contours, the integral over Gamma is 4pi i f(0) = -pi i, not 2pi i f(0).

Question 8:

(a) No solution given. (The curves are straight lines; join the endpoints together to sketch them).

(b) The expression in the question is the same as int_Gamma f(z) dz, where Gamma = Gamma_1 - Gamma_3 - Gamma_2 (note the order) is a simple closed contour, traversing a certain triangle once in the negative direction.

(Solution A) If f had an anti-derivative on Gamma, then the expression would be 0 by the fundamental theorem of calculus.

(Solution B) If f was analytic on Gamma and inside Gamma, then the expression would be 0 by Cauchy's theorem.

(c) 1/(z-1) does not have an anti-derivative on Gamma, and is not analytic inside Gamma, so we can't use (b) directly and say that the answer is 0. If Gamma was **positively** oriented, then the answer would be 2 pi i, by Cauchy's theorem; however Gamma is negatively (clockwise) oriented, so the answer is actually - 2 pi i.

Question 9:

(a) Log(2z-1) will work; it is analytic except when 2z-1 is zero or a negative real, i.e. it is analytic unless z = x + 0i for some x <= 1/2. In particular, it is analytic on gamma. Several other branches also work: L_0(2z-1), however, does not.

Note that Log(2z-1) is not the same thing as Log(z).

(b) This integral can be worked out by direct parameterization, but you have to remember that the anti-derivative of something like 2i/(1-i + 2it) is not ln|1 - i + 2it|, but is more like Log(1 - i + 2it) on gamma. (There is more than one possible anti-derivative for this integrand). The easier way is to use (a) and note that Log(2z-1) is an anti-derivative of 2/(2z-1). Thus the answer is Log(2z-1) evaluated at 1-i/2 and 1+i/2, which works out to pi i /2.

Question 10:

(a) Write the integrand as f(z) / (z-1/2)^2 where f(z) = exp(z^2)/4, and apply the GCIF to get 2 pi i f'(1/2) = pi i e^{1/4} / 2.

(b) This integral is bad at two places inside the contour, so you can't use the CIF directly. Instead, you should either use partial fractions or break up the contour into two smaller closed contours (by adding and subtracting some chord), such that one contour goes around 1 and the other goes around 3. Applying the CIF to each one seperately gives 2pi i sin 1 / (-2) and 2pi i sin 3 / 2 respectively, so the net integral is pi i (sin 3 - sin 1).

(c) e^{i Re z} has magnitude 1, and 1/z^2 has magnitude 1/100 on Gamma, so the integrand has a magnitude of at most 1/100. Since the contour has length 2 pi 10, the integrand has a size of at most (2 pi 10)/100 = pi / 5.

Question 11:

(a) The ratio of consecutive terms in this series converges to |z-1|/2, so the series converges when |z-1| <2 and diverges when |z-1|>2. Thus the disk of convergence is |z-1| < 2.

(b) We want to write 2/(3-z) in terms of powers of (z-1):

```
2/(3-z) = 2/(3 - (z-1+1)) = 2/(2 - (z-1)) = 1/(1 - (z-1)/2)

= 1 + (z-1)/2 + (z-1)^2/4 + (z-1)^3/8 + ...
```

and this expansion is valid when |(z-1)/2| < 1, i.e. when |z-1| < 2.

(c) Differentiating what we have for (b), we obtain

```
2/(3-z)^2 = 1/2 + 2(z-1)/4 + 3(z-1)^2/8 + ...
```

and multiplying by 2 we obtain

```
4/(3-z)^2 = 1 + 2(z-1)/2 + 3(z-1)^2/4 + ... = f(z).
```

Thus f(z) = 4/(3-z)^2.