Outline for Hour Exam II

The short answer to what will be tested on the second hour exam is Sections 2.2-2.6 in Fisher. However, I have skipped around, omitted stuff and added stuff to this part of Fisher's book. Here are few details.

Section 2.2: The ratio test, $1/R = \lim_{n\to\infty} |a_{n+1}/a_n|$ when the limit exists, has been the main tool for finding radii of convergence in this course. Know how to use it. The root test, $1/R = \limsup_{n\to\infty} |a_n|^{1/n}$, is more powerful, and *always* tells you the radius of convergence, but it involves the "limsup" which is not discussed before you take Math 131A. So I left it out. I postponed discussion of derivatives of power series until we had the Cauchy Integral Formula (see Section 2.4). I also did not discuss multiplication and division of power series, but to find the power series for $(a_0 + a_1 z + a_2 z^2 + \cdots)(b_0 + b_1 z + b_2 z^2 + \cdots)$ you just multiply them out and collect terms with same powers of z. Division is worse. The most useful power series is the geometric series $(1-z)^{-1} = \sum_{n=0}^{\infty} z^n$ when $|z| < 1$.

Section 2.3: Cauchy's Theorem is very important. The only part of Fisher's presentation that I changed was the discussion of simple closed curves: instead of talking about $\int_\gamma f(z)dz$ when γ was a piecewise smooth simple closed curve, I just took γ to be the boundary of a domain where we could use Green's theorem, and assumed that γ was "positively oriented": as you move around γ, the domain stays on your left.

At this point I skipped to Cauchy's Formula (Theorem 4) and derived it as Fisher does, though again I replaced the piecewise smooth simple closed curve γ with the boundary of a domain where we can use Green's theorem. This has the advantage that we can immediately use Cauchy's theorem in domains with holes in them (domains that are not "simply connected"). I did not discuss Examples 8 and 9, but there was a homework problem (Section 2.3 #10) like Example 10.

Section 2.4: For this section there are the cryptic notes "Cauchy's Theorem and Power Series". I strongly recommend staring at them until they make sense. They contain both the results on power series of analytic functions (pages 123-124 in Fisher) and on differentiation of power series (pages 97-99 in Fisher). They also make it easy to see why an analytic function $f(z)$ can be written as $f(z) = (z - z_0)h(z)$ with $h(z)$ analytic when f has a zero of order m at z_0. These are the results from Section 2.4 that we will need in the rest of the course. Morera's Theorem was not covered.

Section 2.5: There are three main topics here: (i) Laurent series for functions analytic in domains of the form $r < |z - z_0| < R$, (ii) classification of isolated singularities and (iii) computation of residues at poles. All of these were covered.

Section 2.6: All of this section is devoted to computation of definite integrals of (usually real-valued) functions using residues. There were basically three types f integrals, each requiring a slightly different method

$$(i) \quad \int_0^{2\pi} G(\cos\theta, \sin\theta)d\theta \qquad (ii) \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}dx \qquad \int_{-\infty}^{\infty} \frac{e^{ix}}{P(x)}dx,$$

where $P(x)$ and $Q(x)$ were polynomials and the degree of P was at least two more than the degree of Q. There was a brief discussion of an integral involving fractional powers of x, but that sort of problem is off-limits for the hour exam.

Some problems from old exams for review

1. Consider the power series for $(z-2)^2(1+2z)^{-1}$ about $z=2$. Without doing any calculation give the radius of convergence for this power series (justify your answer). Next find the power series.

2. Evaluate the following assuming that contours are positively oriented:

$$a) \quad \int_{|z-1-i|=2} \frac{e^z}{(z-1)(z+1)^3}dz \qquad b) \quad \int_{|z+2-i|=2} \frac{e^z}{(z-1)(z+1)^3}dz.$$

3. Compute $\int_0^{2\pi} e^{\cos t}\cos(\sin t)dt$.

4. Compute $\int_0^{\infty}(1+x^2)^{-1}\cos x dx$.

5. Find the Laurent expansion of $f(z) = [z(1-z)(3-z)]^{-1}$ in the region $1 < |z| < 3$ (begin by expanding $f(z)$ in partial fractions). As a check on your answer, note that the coefficient of z^{-1} in this Laurent expansion must be Res (f; 0)+Res(f; 1).

6. Classify the singularities of the following functions at $z=0$. If they are poles, give their order.

$$a) \quad \frac{\sin z - z}{z^2} \qquad b) \quad \frac{1}{e^{z^2}-1} \qquad c) \quad \frac{e^{1/z}}{z^2}.$$

Answers: 1. $R = 5/2$ and $f(z) = \sum_{n=0}^{\infty}(-1)^n \frac{2^n}{5^{n+1}}(z-2)^{n+2}$. 2.(a) $\frac{\pi i e}{4}$ (b) a more difficult one: $-5\pi i(4e)^{-1}$. 3. 2π (what else could it be?) 4. $\frac{\pi}{2e}$ 5. $\sum_{n=0}^{\infty}\frac{-1}{2z^{n+2}} + \sum_{n=0}^{\infty}\frac{-1}{6}\frac{z^{n-1}}{3^n}$ 6. (a) removable (b) pole of order 2 (c) essential.