Syllabus for Math 132 up to the first hour exam

Section 1.1: This section introduces complex numbers. Pay attention to: $\overline{wz} = \overline{w}\,\overline{z}$, $|wz| = |w||z|$, the polar form of a complex number (and its usefulness in solving $z^n = a$), and the relation: $\mathrm{Re}\{w\overline{z}\}$ equals the dot product of w and z, considered as vectors.

Section 1.2: Typical results here are: $|z-a| = |z-b|$ defines the straight line through $(a+b)/2$ perpendicular to $b-a$, $z = (1-t)a + tb,\ t \in [0,1]$, is the line segment going from a to b, and $|z-a| = \rho|z-b|$ defines a circle when $\rho \neq 1$.

Section 1.3: This section defines open, closed, connected and convex sets in the plane. Most of the time we will want functions to be defined on open, connected sets. Fisher calls those sets "domains".

Section 1.4: This section defines $\lim_{n\to\infty} a_n$ and $\sum_{n=1}^{\infty} a_n$ for a sequence of complex numbers $\{a_n\}_{n=1}^{\infty}$. It also defines $\lim_{z\to z_0} f(z)$ and continuity at $z = z_0$ for a function $f(z)$.

Those four sections were preliminary, but you need to know all the definitions. The following three have most of the content of the course so far.

Section 1.5: Everything is this section is based on $e^z = e^{x+iy} = e^x(\cos y + i\sin y)$, and the identity $e^{z+w} = e^w e^z$. e^z can also be defined as $\sum_{z=0}^{\infty} \frac{z^n}{n!}$, but that will come later. Since $e^z = e^{z+2\pi i}$, e^z cannot be one-to-one on any domain large enough to contain both z and $z+2\pi i$, and it does not have an inverse defined on $\mathbb{C}$. However, e^z restricted to the open strip $-\pi < y < \pi$ is one-to one, and it maps the strip onto the plane minus the closed negative real axis. When $z = re^{i\theta}$ with $r > 0$ and $-\pi < \theta < \pi$, Log z is defined by Log $z = \ln r + i\theta$, and $z =$ Log w is the unique solution to $e^z = w$ with z in the strip above. In other words Log is the inverse function for e^z restricted to the strip.

Using e^z, one defines $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$ and $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$. Note that e^z, $\cos z$ and $\sin z$ all become the familiar functions e^x, $\cos x$ and $\sin x$ for z on the real axis. In that sense they are *extensions* of those functions to the complex plane. $\sin z$ maps the vertical strip $-\pi/2 < x < \pi/2$ one-to-one onto the plane minus the two slits on the real axis defined by $|x| \geq 1$, and $\cos z$ maps the the vertical strip $0 < x < \pi$ one-to-one onto the plane minus the same two slits. So $\sin z$ and $\cos z$ restricted to the strips have inverse functions defined on the plane minus the slits.

Section 1.6: Part I, Line Integrals. Be sure that you understand the computation of the line integral $\int_C f(z)dz$, where C is a curve in the complex plane. Reversing the direction that C is traced multiplies $\int_C f(z)dz$ by -1, but other than that, $\int_C f(z)dz$ does not depend on the parametrization: one can always use arc-length along C to get a parametrization $z(s)$, $0 \leq s \leq L$, where L is the length of C. If $w(t)$, $a \leq t \leq b$, also traces C, then the function $s(t)$ is defined by $w(t) = z(s(t))$ and satisfies $s(a) = 0$ and $s(b) = L$. Then, just by the chain rule for differentiation, followed by change of variables formula for integrals, you get

$$\int_C f(z)dz \text{ (computed using } w(t)) = \int_a^b f(w(t))\frac{dw}{dt}(t)dt$$

$$= \int_a^b f(z(s(t))\frac{dz}{ds}(s(t))\frac{ds}{dt}(t)dt$$

$$= \int_0^L f(z(s))\frac{dz}{ds}(s)ds = \int_C f(z)dz \text{ (computed using } z(s)).$$

Fisher derives the following useful lemma: if f is a continuous function defined for $|z-a| < \delta$, and $C(\epsilon)$ is the circle traced by $z = a + \epsilon e^{it}$, $0 \le t < 2\pi$, then

$$\lim_{\epsilon \to 0} \int_{C(\epsilon)} \frac{f(z)}{z-a} dz = 2\pi i f(a). \quad (1)$$

This section also contains the basic estimate $|\int_C f(z)dz| \le ML$ where M is the maximum value of $|f(z)|$ on C and L is the length of C.
Part II, Green's Theorem. Know the statement of Green's Theorem and the proof of the following corollary: Assume that u and v are real-valued functions with continuous partial derivatives in the domain D. If Γ is the boundary of D traced so that D is on the left as you follow Γ, then

$$\int_\Gamma (u+iv)(dx+idy) = -\int_D (\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})dxdy + i\int_D (\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y})dxdy. \quad (2)$$

Section 2.1: The definition of the *complex* derivative $f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$ is here. Please remember the following theorem: If $u(x,y)$ and $v(x,y)$ are differentiable functions of (x,y), then the complex derivative of $f = u+iv$ exists if and only if u and v satisfy the Cauchy-Riemann equations, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, and f' is given by

$$f'(x+iy) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) = \frac{1}{i}(\frac{\partial u}{\partial y}(x,y) + i\frac{\partial v}{\partial y}(x,y)).$$

Assuming that u and v have continuous partial derivatives in a domain with boundary Γ and satisfy the Cauchy-Riemann equations, you can combine (1) and (2) to get "The Cauchy Integral formula" for $f = u+iv$

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w-z}dw.$$

This will be *very* important in the rest of the course, but will not be used in this first hour exam.

Functions which have complex derivatives at all points of a connected, open set D in the plane are called "analytic" functions on the domain D. When functions have complex derivatives, all the usual rules of differentiation hold: $(f+g)' = f' + g'$, $(fg)' = gf' + fg'$, $(f/g)' = (gf' - fg')/g^2$ and $(f(g))'(z) = f'(g(z))g'(z)$.

If u and v satisfy the Cauchy-Riemann equations in a domain D, there are the following consequences:

(a) Both u and v are "harmonic" in D: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ in D. Strictly speaking this requires that u and v have second partial derivatives, but we will see that, as soon as $u+iv$ is analytic in D, u and v have derivatives of all orders in D.

(b) The function u determines the gradient of v, and hence determines v up to an additive constant. Likewise v determines the gradient of u, and hence determines u up to an additive constant. u and v are called "conjugate harmonic functions".