Name: ______________________________

SID#: ______________________________

Directions:

Do not open this exam until the professor instructs you to do so.

Turn off your pagers and cell-phones.

You have 50 minutes to complete the midterm exam.

When instructed to begin, solve the following problems in the space provided.

Be sure to justify any claims you make.

You may cite without proof any facts that were either assigned as homework problems or covered in lecture unless the facts are explicitly stated as midterm problems.

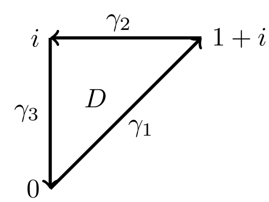
You may not consult any books, notes, exams, quizzes, students, computers, pagers, cell phones, or any other aids during the exam.

Be sure to put your SID # on the top right corner of every page.

Relax. Do the best you can. Think before you write. Write neatly.

	Points	Possible Points
Problem 1	10	10
Problem 2	10	10
Problem 3	10	10
Problem 4	10	10
Problem 5	10	10
Total Score:	50	50

1. (a) Consider the following closed contour ∂D made up of paths γ_1, γ_2, and γ_3.



Let $dz = dx + idy$. Evaluate the following integral using any method.

$$\int_{\partial D} \text{Im}\, z\ dz$$

Solution.

$$\text{Im}\, z\ dz = y(dx + idy) = ydx + iydy$$

Thus by Green's Theorem,

$$\begin{aligned}
\int_{\partial D} \text{Im}\, z\ dz &= \int_{\partial D} ydx + iydy \\
&= \iint_D \left(\frac{\partial (iy)}{\partial x} - \frac{\partial y}{\partial y} \right) dxdy \\
&= \iint_D (0-1) dxdy \\
&= -\iint_D dxdy \\
&= -Area(D) \\
&= -\frac{1}{2}
\end{aligned}$$

Alternatively, parametrize ∂D by

$$\begin{aligned}
\gamma_1(t) &= t(i+1) && \text{for} \quad 0 \le t \le 1 \\
\gamma_2(t) &= (i+1) - t && \text{for} \quad 0 \le t \le 1 \\
\gamma_3(t) &= (1-t)i && \text{for} \quad 0 \le t \le 1
\end{aligned}$$

Thus

$$\begin{aligned}
\int_{\partial D} \text{Im}\, z\ dz &= \int_{\gamma_1} \text{Im}\, z\ dz + \int_{\gamma_2} \text{Im}\, z\ dz + \int_{\gamma_3} \text{Im}\, z\ dz \\
&= \int_0^1 t(1+i)dt + \int_0^1 1(-1)dt + \int_0^1 (1-t)(-i)dt \\
&= (1+i)\frac{t^2}{2}\bigg|_0^1 + (-1)t|_0^1 + (-i)\left(t - \frac{t^2}{2}\right)\bigg|_0^1 \\
&= \frac{1+i}{2} - 1 - \frac{i}{2} \\
&= -\frac{1}{2}
\end{aligned}$$

(b) Is the the differential $\operatorname{Im} z\, dz$ exact, closed, or neither on $\mathbb{C}$? Explain.

Solution. *Since* $\operatorname{Im} z$ *is continuously (real) differentiable on the star shaped domain* $\mathbb{C}$*, exact is equivalent to closed. It is not closed on* $\mathbb{C}$ *since it is not analytic on* $\mathbb{C}$*. Alternatively, by definition, it is not closed on* $\mathbb{C}$ *since*

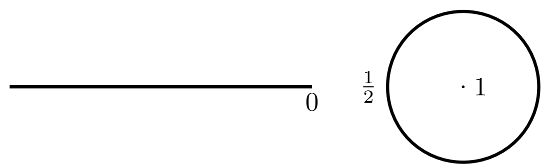
$$\left(\frac{\partial(iy)}{\partial x} - \frac{\partial y}{\partial y}\right) = -1 \neq 0$$

It is not exact on $\mathbb{C}$ *since exact implies closed. Alternatively, we showed that the integral in (a) is nonzero and hence not path independent on* ∂D*. Hence, not exact.*

2. Evaluate the following integrals.

(a) $\displaystyle \frac{1}{2\pi i}\oint_{|z-1|=\frac{1}{2}} \sqrt{z}\, dz$ where $\sqrt{z} = e^{\frac{1}{2}\operatorname{Log}(z)}$ is the principal square root on $\mathbb{C}\backslash(-\infty, 0]$.

Solution. *As the picture suggests,* $\sqrt{z}$ *is analytic on* $|z-1| \leq \frac{1}{2}$*:*



Thus,

$$\oint_{|z-1|=\frac{1}{2}} \sqrt{z}\, dz = 0$$

by Cauchy's Theorem or by the Fundamental Theorem of Calculus for Analytic Functions:

$$\oint_{|z-1|=\frac{1}{2}} \sqrt{z}\, dz = \int_{\frac{1}{2}}^{\frac{1}{2}} \sqrt{z}\, dz = \frac{2}{3} z^{\frac{3}{2}}\Big|_{\frac{1}{2}}^{\frac{1}{2}} = 0$$

(b) $\displaystyle \frac{1}{2\pi i}\oint_{|z|=3} \frac{e^{2z}}{z-1}\, dz$

Solution. *By the Residue Therorem or by Cauchy's Integral Formula,*

$$\frac{1}{2\pi i}\oint_{|z|=3} \frac{e^{2z}}{z-1}\, dz = e^2$$

3. (a) Use power series to find a complex antiderivative $F(z)$ of the following function.

$$f(z) = \frac{z}{e^z}$$

Solution. *For all z, we have*

$$e^{-z} = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} = 1 + (-z) + \frac{(-z)^2}{2!} + \frac{(-z)^3}{3!} + \frac{(-z)^4}{4!} \cdots$$

$$ze^{-z} = z \left[\sum_{n=0}^{\infty} \frac{(-z)^n}{n!} \right] = z - z^2 + \frac{z^3}{2!} - \frac{z^4}{3!} + \frac{z^5}{4!} + \cdots$$

Thus f has the following Taylor expansion about 0:

$$f(z) = ze^{-z} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{n+1}}{n!}$$

Therefore

$$\begin{aligned} F(z) &= \int_0^z we^{-w} dw \\ &= \int_0^z \sum_{n=0}^{\infty} \frac{(-1)^n w^{n+1}}{n!} dw \\ &= \sum_{n=0}^{\infty} \int_0^z \frac{(-1)^n w^{n+1}}{n!} dw \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{n+2}}{(n+2)n!} \\ &= \sum_{n=2}^{\infty} \frac{(-1)^n z^n}{n(n-2)!} \end{aligned}$$

(b) What is the radius of convergence of the Taylor series centered at 0 for $F(z)$? Explain.

Solution. *The radius of convergence $R = \infty$ since integrating a power series does not change its radius of convergence and ze^{-z} has $R = \infty$.*
Alternatively,

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(n+1)(n-1)!} \frac{n(n-2)!}{(-1)^n} \right| = \lim_{n\to\infty} \left| \frac{n}{(n+1)(n-1)} \right| = 0$$

Thus $R = \infty$.

4. Determine whether the following isolated singularities at z_0 are removable, poles, or essential. To receive full credit, be sure to justify your answer. If a pole, be sure to also specify its order.

(a) $\sin\left(\frac{1}{z}\right), \quad z_0 = 0$

Solution. *Since the Laurent series expansion for* $0 < |z| < \infty$ *is*

$$\sin\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{-2n-1}}{(2n+1)!} = \sum_{n=-\infty}^{0} \frac{(-1)^n z^{2n-1}}{(-2n+1)!}$$

we see that 0 *is an essential singularity since* $a_n = \frac{(-1)^n}{(-2n+1)!} \neq 0$ *for infinitely many* $n < 0$ *(all of them).*

(b) $2z^2 + 1, \quad z_0 = \infty$

Solution. *Since*

$$g(w) = f(w^{-1}) = \frac{2}{w^2} + 1$$

has a double pole at $w = 0$ *(by the Laurent expansion you see above for* $g(w)$ *on* $0 < |w| < \infty$*), we see that* $f(z)$ *has a double pole at* ∞.

(c) $\frac{e^z - 1}{z}, \quad z_0 = 0$

Solution. *Since*

$$\begin{aligned} e^z - 1 &= -1 + 1 + z + \frac{z^2}{2!} + \cdots = z + \frac{z^2}{2!} + O(z^3) \\ \frac{e^z - 1}{z} &= 1 + z + \frac{z}{2!} + O(z^2) \end{aligned}$$

we see that

$$\lim_{z \to 0} \frac{e^z - 1}{z} = 1$$

Thus

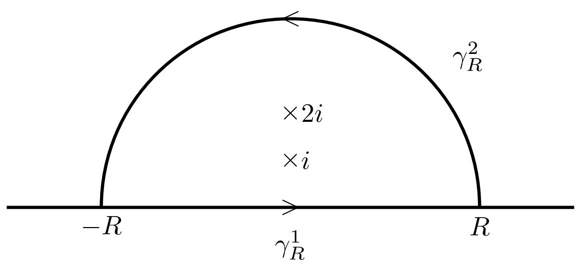
$$\frac{e^z - 1}{z}$$

is bounded near 0. *Thus* 0 *is a removable singularity by Riemann's Removable Singularities Theorem.*

5. Use Residue Theory and a semicircular contour to show

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)} = \frac{\pi}{6}$$

Solution. *Consider the semicircular contour* Γ_R*:*



The contour Γ_R

The function

$$f(z) = \frac{1}{(z^2+1)(z^2+4)} = \frac{1}{(z+i)(z-i)(z+2i)(z-2i))} = \frac{1}{z^4+5z^2+4}$$

has simple poles at $i, 2i$ *inside* Γ_R*. The other two poles* $-i$ *and* $-2i$ *are in the lower half plane and thus outside the contour. Thus for large* $R > 2$*, we have by the Residue Theorem:*

$$\int_{\Gamma_R} f(z)dz = 2\pi i \left[\text{Res}\left[f(z), i\right] + \text{Res}\left[f(z), 2i\right]\right]$$

and by Rule 3 (Rule 4)

$$\begin{aligned}
\text{Res}\left[f(z), i\right] &= \left.\frac{1}{4z^3+10z}\right|_{z=i} = \frac{1}{4i^3+10i} = \frac{1}{6i} \\
\text{Res}\left[f(z), -i\right] &= \left.\frac{1}{4z^3+10z}\right|_{z=2i} = \frac{1}{4(2i)^3+20i} = \frac{1}{-32i+20i} = \frac{-1}{12i}
\end{aligned}$$

By an ML *estimate, on* γ_R^2*, we have* $|z| = R$ *and thus*

$$\left|\int_{\gamma_R^2} \frac{1}{z^2+1z^2+4} dz\right| \leq \frac{1}{(R^2-1)(R^2-4)} \pi R \to 0 \text{ as } R \to \infty$$

Therefore for all $R > 2$

$$\int_{-R}^{R} \frac{dx}{(x^2+1)(x^2+4)} + \int_{\gamma_R^2} \frac{dz}{z^2+1z^2+4} = 2\pi i\left[\frac{1}{6i} - \frac{1}{12i}\right] = \frac{\pi}{6}$$

and taking limits as $R \to \infty$*, we have*

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)} + 0 = \frac{\pi}{6}$$