Math 132 Complex Analysis for Applications | Midterm I Solution

Name: Key

SID#:

Directions:

Do not open this exam until the professor instructs you to do so.

Turn off your pagers and cell-phones.

You have 50 minutes to complete the midterm exam.

When instructed to begin, solve the following problems in the space provided.

Be sure to justify any claims you make.

You may cite without proof any facts that were either assigned as homework problems or covered in lecture unless the facts are explicitly stated as midterm problems.

You may not consult any books, notes, exams, quizzes, students, computers, pagers, cell phones, or any other aides during the exam.

Be sure to put your SID # on the top right corner of every page.

Relax. Do the best you can. Think before you write. Write neatly.

	Points	Possible Points
Problem 1	10	10
Problem 2	10	10
Problem 3	10	10
Problem 4	10	10
Problem 5	10	10
Total Score:	50	50

1. Let z_1 and z_2 be nonzero complex numbers such that $\theta = \arg(z_2/z_1)$ is the angle between them.

   (a) The **dot product** $z_1 \circ z_2$ is defined as $\text{Re}(\overline{z_1}z_2)$. Show that

$$z_1 \circ z_2 = |z_1||z_2| \cos\theta$$

   (b) The **cross product** $z_1 \times z_2$ is defined as $\text{Im}(\overline{z_1}z_2)$. Show that

$$z_1 \times z_2 = |z_1||z_2| \sin\theta$$

Solution. *Using the Polar Representations* $z_1 = |z_1|e^{i\theta_1}$ *and* $z_2 = |z_2|e^{i\theta_2}$

$$\begin{aligned}
z_1 \circ z_2 + i(z_1 \times z_2) &= \overline{z_1}z_2 \\
&= |z_1|e^{-i\theta_1}|z_2|e^{i\theta_2} \\
&= |z_1||z_2|e^{i(\theta_2-\theta_1)} \\
&= |z_1||z_2|e^{i\theta} \\
&= |z_1||z_2|\cos\theta + i|z_1||z_2|\sin\theta
\end{aligned}$$

Equating real and imaginary parts in the equation

$$z_1 \circ z_2 + i(z_1 \times z_2) = |z_1||z_2|\cos\theta + i|z_1||z_2|\sin\theta$$

we have proved (a) and (b).

2. (a) Find the cube roots of $8i$. Plot them on the complex plane.

Solution. *The cube roots of* $8i$ *are the cube roots of* 1

$$\{1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}\}$$

multiplied by the principal cube root of $8i$*:*

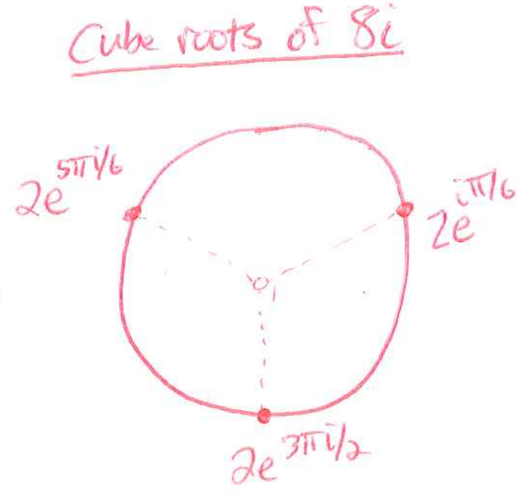
$$|8i|^{\frac{1}{3}} e^{\frac{i\mathrm{Arg}(8i)}{3}} = 8^{\frac{1}{3}} e^{\frac{i\pi}{6}} = 2e^{\frac{i\pi}{6}}$$

Thus the cube roots of $8i$ *are*

$$\{2e^{\frac{i\pi}{6}}, 2e^{\frac{i\pi}{6}} e^{\frac{2\pi i}{3}}, 2e^{\frac{i\pi}{6}} e^{\frac{4\pi i}{3}}\}$$

which simplifies to

$$\{2e^{\frac{i\pi}{6}}, 2e^{\frac{5\pi i}{6}}, 2e^{\frac{3\pi i}{2}}\}$$



(b) Compute all values of $\left(\sqrt{3}+i\right)^i$.

Solution.

$$\begin{aligned}\left(\sqrt{3}+i\right)^i &= \{e^{i\left(\log|\sqrt{3}+i| + i\mathrm{Arg}(\sqrt{3}+i) + 2\pi i k\right)} \mid k = 0, \pm1, \pm2, \dots\} \\ &= \{e^{i\log 2 - \frac{\pi}{6} - 2\pi k} \mid k = 0, \pm1, \pm2, \dots\} \\ &= \{e^{-\frac{\pi}{6} - 2\pi k + i\log 2} \mid k = 0, \pm1, \pm2, \dots\}\end{aligned}$$

3. (a) Show that the following function is harmonic.

$$u(x,y) = x^2 - y^2 - 2xy$$

Solution.

$$\frac{\partial u}{\partial x} = 2x - 2y$$
$$\frac{\partial u}{\partial y} = -2y - 2x$$
$$\frac{\partial^2 u}{\partial x^2} = 2$$
$$\frac{\partial^2 u}{\partial y^2} = -2$$

We have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$$

Thus u is harmonic.

(b) Find a harmonic conjugate $v(x,y)$ for u.

Solution. *Since $u + iv$ must be analytic, by the Cauchy-Riemman equations*

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x - 2y$$
$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2x + 2y$$

The harmonic conjugate is given by

$$\begin{aligned}
v(x,y) &= -\int_{x_0}^{x} \frac{\partial u}{\partial y}(s, y_0)ds + \int_{y_0}^{y} \frac{\partial u}{\partial x}(x,t)dt \\
&= \int_{x_0}^{x} (2s + 2y_0)ds + \int_{y_0}^{y} (2x - 2t)dt \\
&= \int_{x_0}^{x} (2s)ds + 2y_0(x - x_0) - \int_{y_0}^{y} 2tdt + 2x(y - y_0) \\
&= s^2\Big|_{x_0}^{x} + 2y_0(x - x_0) - t^2\Big|_{y_0}^{y} + 2x(y - y_0) \\
&= x^2 - (x_0)^2 - 2x_0y_0 - y^2 + y_0^2 + 2xy \\
&= x^2 - y^2 + 2xy - [x_0^2 - y_0^2 + 2x_0y_0]
\end{aligned}$$

where (x_0, y_0) is a fixed point in $\mathbb{C}$.

4. Consider the following complex-valued function.

$$f(z) = z + a\overline{z}$$

Prove using the definition of complex derivative that $f(z)$ is analytic if and only if $a = 0$.

$$\begin{aligned} f'(z) &= \lim_{h \to 0} \frac{(z+h) + a(\overline{z+h}) - (z + a\overline{z})}{h} \\ &= \lim_{h \to 0} \frac{z + h + a\overline{z} + a\overline{h} - z - a\overline{z}}{h} \\ &= \lim_{h \to 0} \frac{h + a\overline{h}}{h} \end{aligned}$$

If we let h be real as it approaches 0,

$$\lim_{h \to 0} \frac{h + a\overline{h}}{h} = \lim_{h \to 0} \frac{(1+a)h}{h} = 1 + a$$

If we let h be purely imaginary as it approaches 0,

$$\lim_{h \to 0} \frac{h + a\overline{h}}{h} = \lim_{h \to 0} \frac{(1-a)h}{h} = 1 - a$$

In order for this limit to exist, $1 + a = 1 - a$. This occurs if and only if $a = 0$.

5. Consider the following fractional linear transformation (FLT).

$$f(z) = \frac{z+1}{z-1}$$

(a) Determine the image of the unit circle under f. Sketch the unit circle in the z-plane and its image under f in the w-plane.

Solution. *Since*

$$1 \to \infty$$
$$i \to \frac{i+1}{i-1} = -i$$
$$-1 \to 0$$

the unit circle maps to the imaginary axis $i\mathbb{R}$ *under* f*, since FLT's map (lines and circles) to (lines and circles) and three points uniquely determine a generalized circle on* $\mathbb{C} \cup \{\infty\}$.

(b) Determine the image of the open unit disk $\{z : |z| < 1\}$ under f? Sketch the open unit disk in the z-plane and its image under f in the w-plane.

Solution. *Since FLTs preserve orientation and the unit disk is to the left of the unit circle traversed counter clockwise, the image of the unit disk is to the left of the image of the unit circle which moves up the imaginary axis. Thus the image of the unit disk is the left half-plane* $\operatorname{Re} w < 0$.