Mathematics 132/3
Yiannis N. Moschovakis
Test #2, March 7, 2012

There are 100 points available in the test, as indicated, and there is plenty of working space on the back of each page as well as a blank page at the end.

The test is a bit long, and some parts are more difficult than others: do first those parts that you can do without much thinking, and come back afterwards to work on the rest.

All contours are positively oriented!

Good luck!

Solved test

Name: ____________________

Signature: ____________________

Problem 1. __________

Problem 2. __________

Problem 3. __________

Problem 4. __________

Problem 5. __________

Problem 6. __________

Total: ____________________

Problem 1 (10 pts). Compute the following integrals:

(1.1) $\displaystyle\int_{|z-3i|=1} \frac{\cos z dz}{z-3i}$

Solution. We use the Cauchy Formula for $\cos z$ at the point $3i$ with the contour $|z-3i|=1$,

$$\int_{|z-3i|=1} \frac{\cos z dz}{z-3i} = 2\pi i \cos(3i).$$

For a neater expression of the answer, we compute:

$$\cos(3i) = \frac{1}{2}\left(e^{i3i} + e^{-i3i}\right) = \frac{1}{2}\left(e^{-3} + e^{3}\right).$$

Ans. $\boxed{\pi i\left(e^{-3} + e^{3}\right)}$

(1.2) $\displaystyle\int_{|z|=4} \frac{\cos z dz}{z-3i}$

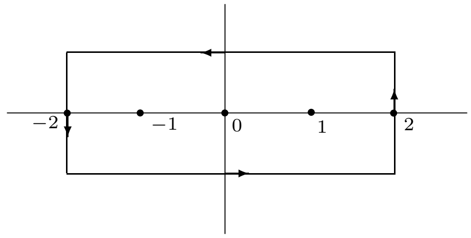
Solution. Same method and value, the point $3i$ is not at the center of the circle $|z|=4$ this time, but still within the contour.

Ans. $\boxed{\pi i\left(e^{-3} + e^{3}\right)}$

(1.3) $\displaystyle\int_{|z|=1} \frac{\cos z dz}{z-3i}$

Solution. This time the singularity lies outside the contour $|z|=1$, and so the integral is 0 by Cauchy's Theorem.

Ans. $\boxed{0}$

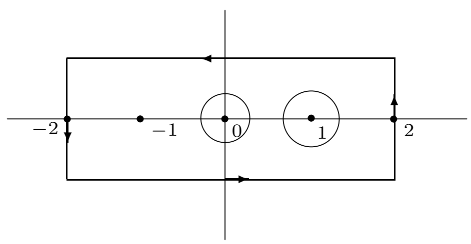


Problem 2 (20 pts). Compute the integral

$$\int_\Gamma \frac{dz}{z(z-1)^2(z+3)}$$

where Γ is the contour in the figure.

Solution.



There are just two singularities of the integrand within Γ, 0 and 1. We draw two positively oriented little circles $|z| = \frac{1}{3}$ and $|z-1| = \frac{1}{3}$ around 0 and 1, so that they are within the contour Γ and they do not intersect. We then appeal to the Annulus Theorem and to the Cauchy formulas for a function and its derivative as follows:

$$\begin{aligned}
\int_\Gamma \frac{dz}{z(z-1)^2(z+3)} &= \int_{|z|=\frac{1}{3}} \frac{dz}{z(z-1)^2(z+3)} + \int_{|z-1|=\frac{1}{3}} \frac{dz}{z(z-1)^2(z+3)} \\
&= \int_{|z|=\frac{1}{3}} \frac{\frac{1}{(z-1)^2(z+3)}dz}{z} + \int_{|z-1|=\frac{1}{3}} \frac{\frac{1}{z(z+3)}dz}{(z-1)^2} \\
&= 2\pi i\Big[\frac{1}{(z-1)^2(z+3)}\Big]_{z=0} + 2\pi i\Big[\frac{d}{dz}\Big(\frac{1}{z^2+3z}\Big)\Big]_{z=1} \\
&= 2\pi i\frac{1}{3} + 2\pi i\Big[\frac{-2z-3}{(z^2+3z)^2}\Big]_{z=1} \\
&= \frac{2}{3}\pi i + 2\pi i\frac{-2-3}{(1+3)^2} = \frac{2}{3}\pi i - 2\pi i\frac{5}{16} = \frac{2}{3}\pi i - \frac{5}{8}\pi i = \frac{\pi i}{24}.
\end{aligned}$$

Ans. $\boxed{\frac{\pi i}{24}}$

Problem 3 (20 pts).

Let

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{3^k} z^{k+1} \quad (|z| < R),$$

where R is the radius of convergence of this power series.

(3.1) What is R?

Solution.

$$\lim_{k\to\infty} \frac{\frac{1}{3^{k+2}}}{\frac{1}{3^{k+1}}} = \lim_{k\to\infty} \frac{3^{k+1}}{3^{k+2}} = \frac{1}{3},$$

so $\boxed{R = 3}$.

Ans. $\boxed{R = 3}$

(3.2) What is the power series expansion of the derivative $f'(z)$, for $|z| < R$?

Solution. We have

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{3^k} z^{k+1} = z + \frac{1}{3} z^2 + \frac{1}{3^2} z^3 + \frac{1}{3^3} z^4 + \cdots,$$

so differentiating term-by-term (which works in the disk of convergence),

$$f'(z) = 1 + \frac{2}{3} z + \frac{3}{3^2} z^2 + \frac{4}{3^3} z^3 + \cdots = \sum_{k=0}^{\infty} \frac{k+1}{3^k} z^k.$$

Ans. $\boxed{f'(z) = \sum_{k=0}^{\infty} \frac{k+1}{3^k} z^k}$

Problem 3 continues on the next page

Problem 3 continues from the previous page

(3.3) Compute:

(i) $f'(0) =$

(ii) $f''(0) =$

Solution. Using the power series for the derivative above, we get

$$f'(0) = 1;$$

and if we differentiate the power series for the derivative we get

$$f''(z) = \frac{2}{3} + \frac{2 \cdot 3}{3^2} z + \cdots ,$$

from which we get $f''(0) = \frac{2}{3}$.

Ans. $\boxed{f'(0) = 1,\ f''(0) = \frac{2}{3}}$

(3.4) Can you find an explicit formula for $f(z)$?

Solution.

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{3^k} z^{k+1} = z \sum_{k=0}^{\infty} \left(\frac{z}{3}\right)^k = z \frac{1}{1 - \frac{z}{3}} = \frac{3z}{3 - z}.$$

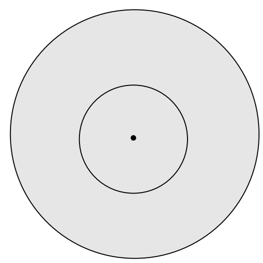
Ans. $\boxed{f(z) = \dfrac{3z}{3 - z}}$

Problem 4 (20 pts). Find the Laurent expansion of the function

$$f(z) = \frac{1}{z-i} + \frac{1}{(z-2)^3}$$

in (positive and negative) powers of z, in the annulus $1 < |z| < 2$.

Solution. Within the big circle $|z| < 2$:



$$\frac{1}{2-z} = \frac{1}{2}\frac{1}{1-\frac{z}{2}} = \frac{1}{2}\Big(1 + \frac{z}{2} + \frac{z^2}{2^2} + \cdots\Big) = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} z^k.$$

Differentiating both sides,

$$\frac{1}{(2-z)^2} = \sum_{k=1}^{\infty} \frac{k}{2^{k+1}} z^{k-1} = \sum_{k=0}^{\infty} \frac{k+1}{2^{k+2}} z^k.$$

Differentiating again,

$$\frac{2}{(2-z)^3} = \sum_{k=1}^{\infty} \frac{(k+1)k}{2^{k+2}} z^{k-1} = \sum_{k=0}^{\infty} \frac{(k+2)(k+1)}{2^{k+3}} z^k,$$

and changing signs and dividing by 2,

$$\frac{1}{(z-2)^3} = \boxed{-\sum_{k=0}^{\infty} \frac{(k+2)(k+1)}{2^{k+4}} z^k}.$$

Outside the small circle $|z| > |i| = 1$,

$$\frac{1}{z-i} = \frac{1}{z}\Big(\frac{1}{1-\frac{i}{z}}\Big) = \frac{1}{z}\Big(1 + \frac{i}{z} + \frac{i^2}{z^2} + \cdots\Big) = \boxed{\sum_{k=1}^{\infty} i^{k-1} \frac{1}{z^k}}.$$

(You can compute here what i^{k-1} is depending on the remainder of $k-1$ by 4, but it does not make for a cleaner answer.)

The Laurent series for $f(z)$ in the annulus is the sum of these two:

$$f(z) = -\sum_{k=0}^{\infty} \frac{(k+2)(k+1)}{2^{k+4}} z^k + \sum_{k=1}^{\infty} i^{k-1} \frac{1}{z^k}.$$

Ans. $\boxed{f(z) = -\sum_{k=0}^{\infty} \frac{(k+2)(k+1)}{2^{k+4}} z^k + \sum_{k=1}^{\infty} i^{k-1} \frac{1}{z^k}.}$

Problem 5 (10pt). You may use in this problem the power series for $\sin z$ about 0,

$$\sin z = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \cdots$$

(5.1) Let $f(z) = \dfrac{\sin z}{z^3}$.

(a) 0 is an isolated singularity for this function; what kind is it?

Solution. We compute the Laurent series

$$\frac{1}{z^3}\sin z = \frac{1}{z^3}\left(z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \cdots\right) = \frac{1}{z^2} - \frac{1}{3!} + \frac{1}{5!}z^2 + \cdots$$

Ans. $\boxed{\text{0 is a pole (of order 2)}}$

(b) Compute $\displaystyle\int_{|z|=1} \frac{\sin z}{z^3}\, dz$

Solution. Integrating the Laurent series term-by-term we get 0, because none of the terms is z^{-1},

Ans. $\boxed{\displaystyle\int_{|z|=1} \frac{\sin z}{z^3}\, dz = 0}$

There is another part of this problem on the next page

(5.2) Let $g(z) = z^4 \sin\left(\frac{1}{z}\right)$.

(a) 0 is an isolated singularity for this function; what kind is it?

Solution. We compute the Laurent series:

$$z^4 \sin\left(\frac{1}{z}\right) = z^4\Big(\frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \frac{1}{7!z^7} + \cdots\Big) = z^3 - \frac{1}{3!}z + \frac{1}{5!z} - \frac{1}{7!z^3} + \cdots$$

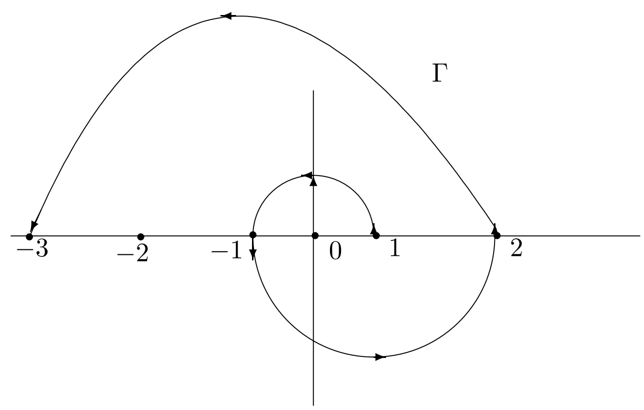
Ans. $\boxed{\text{0 is an essential singularity}}$

(b) Compute $\displaystyle\int_{|z|=1} z^4 \sin\left(\frac{1}{z}\right)\, dz$

Solution. We integrate the Laurent series term-by-term. All the terms yield 0, except the one with the z^{-1} in it, and so we get

$$\int_{|z|=1} z^4 \sin\left(\frac{1}{z}\right)\, dz = \int_{|z|=1} \frac{1}{5!z}\, dz = \frac{2\pi i}{5!} = \frac{\pi i}{60}.$$

Ans. $\boxed{\displaystyle\int_{|z|=1} z^4 \sin\left(\frac{1}{z}\right) = \frac{\pi i}{60}}$



Problem 6 (20pt). Compute the integral

$$\int_\Gamma \frac{dz}{z}$$

where Γ is the contour in the figure, which starts at 1 and ends up at -3, after going around the origin.

Solution. Consider the three contours:

$$\alpha: \text{starts at 1 and ends at } -1 \text{ following } \Gamma,$$
$$\beta: \text{starts at } -1 \text{ and ends at 2 following } \Gamma,$$
$$\gamma: \text{starts at 2 and ends at } -3 \text{ following } \Gamma.$$

Clearly $\Gamma = \alpha + \beta + \gamma$, and so

$$\int_\Gamma \frac{dz}{z} = \int_\alpha \frac{dz}{z} + \int_\beta \frac{dz}{z} + \int_\gamma \frac{dz}{z}.$$

So we need to compute these three integrals separately, and for each of them we need a branch of the logarithm which is analytic on contour and so we can use it as an anrtiderivative of $\frac{1}{z}$. The branches that are needed are easy to find.

$$\int_\alpha \frac{dz}{z} = \log_{-\frac{\pi}{2}}(-1) - \log_{-\frac{\pi}{2}}(1) = \text{Log}(|-1|) + i\pi - (\text{Log}(1) + i \cdot 0) = \boxed{\pi i}$$

$$\int_\beta \frac{dz}{z} = \log_{\frac{\pi}{2}}(2) - \log_{\frac{\pi}{2}}(-1) = \text{Log}(|2|) + i2\pi - (\text{Log}(|-1|) + i\pi) = \boxed{\text{Log}\,2 + \pi i}$$

$$\int_\gamma \frac{dz}{z} = \log_{-\frac{\pi}{2}}(-3) - \log_{-\frac{\pi}{2}}(2) = \text{Log}(|-3|) + i\pi - (\text{Log}\,2 + i \cdot 0) = \boxed{\text{Log}\,3 - \text{Log}\,2 + \pi i}$$

So

$$\int_\Gamma \frac{dz}{z} = \pi i + \text{Log}\,2 + \pi i + \text{Log}\,3 - \text{Log}\,2 + \pi i = \text{Log}\,3 + 3\pi i.$$

Ans. $\boxed{\text{Log}\,3 + 3\pi i}$