Mathematics 132/1
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All parts of each problem are worth (approximately) the same, and there are 110 points altogether as indicated. There is plenty of working space on the back of each page as well as a blank page at the end.

Some parts of each problem are more difficult than others: do first those parts that you can do without much thinking, and come back afterwards to work on the rest.

In your answers, you can leave expressions like $\cos(2), e^{\pi}$, etc. as they are.

Show your work to justify your answers!

Good luck!

Solved test

Name: ____________________

Signature: ____________________

Problem 1. __________

Problem 2. __________

Problem 3. __________

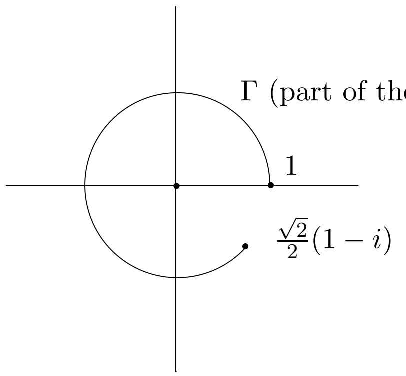
Problem 4. __________

Problem 5. __________

Problem 6. __________

Total: ____________________

Problem 1 (15 points). Compute the following three integrals along the contour Γ, which is part of the positively oriented unit circle.



(1.1) Compute $\int_\Gamma \frac{1}{z}\,dz$.

Solution. Note that $\frac{\sqrt{2}}{2}(1-i) = e^{\frac{7\pi}{4}i}$.

We need an antiderivative of $\frac{1}{z}$ which is analytic on the contour, so we pick

$$\log_{-\frac{\pi}{8}}(z)$$

and we compute:

$$\int_\Gamma \frac{1}{z}\,dz = \log_{-\frac{\pi}{8}}(e^{\frac{7\pi}{4}i}) - \log_{-\frac{\pi}{8}}(1) = \frac{7\pi i}{4}.$$

Ans. $\boxed{\frac{7\pi i}{4}}$

(1.2) Compute $\int_\Gamma \overline{z}\,dz$ ($\overline{z}$ is the conjugate of z).

Solution. When z is on the unit circle,

$$\overline{z} = \frac{\overline{z}z}{z} = \frac{|z|^2}{z} = \frac{1}{z},$$

so this is the same integral as the previous one,

$$\int_\Gamma \overline{z}\,dz = \int_\Gamma \frac{1}{z}\,dz = \frac{7\pi i}{4}.$$

Ans. $\boxed{\frac{7\pi i}{4}}$

(1.3) Compute $\int_\Gamma \frac{1}{z^2}\,dz$

Solution. This integrand has an antiderivative in the entire plane, since

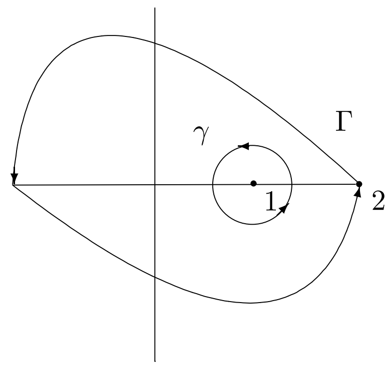
$$\frac{d}{dz}(-\frac{1}{z}) = \frac{1}{z^2}.$$

So

$$\int_\Gamma \frac{1}{z^2} = -\frac{1}{z}\Big|_{\frac{\sqrt{2}}{2}(1-i)}^{1} = -\frac{1}{\frac{\sqrt{2}}{2}(1-i)} + \frac{1}{1} = 1 - \frac{\sqrt{2}(1+i)}{(1-i)(1+i)} = 1 - \frac{\sqrt{2}}{2}(1+i) = 1 - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i.$$

Ans. $\boxed{1 - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i}$

Problem 2 (30 points). Compute the following two integrals along the indicated closed, positively oriented contours.



(2.1) $\oint_\gamma \frac{\cos z}{z(z-1)^2}\, dz$

Solution.

$$\oint_\gamma \frac{\cos z}{z(z-1)^2}\, dz = \oint_\gamma \frac{\frac{\cos z}{z}}{(z-1)^2}\, dz = 2\pi i \Big(\frac{d}{dz}\frac{\cos z}{z}\Big)\Big]_{z=1} =$$

$$2\pi i \frac{z(-\sin z) - \cos z}{z^2}\Big]_{z=1} = -2\pi i(\sin 1 + \cos 1).$$

Ans. $\boxed{-2\pi i(\sin 1 + \cos 1)}$

(2.2) $\oint_\Gamma \frac{\cos z}{z(z-1)^2}\,dz$

Solution. Let γ' be a positively oriented small circle about 0 which misses γ. Then

$$\oint_{\gamma'} \frac{\cos z}{z(z-1)^2}\,dz = \oint_{\gamma'} \frac{\frac{\cos z}{(z-1)^2}}{z}\,dz = 2\pi i \frac{\cos 0}{(0-1)^2} = 2\pi i,$$

and then by the Annulus Theorem,

$$\oint_\Gamma \frac{\cos z}{z(z-1)^2}\,dz = \oint_\gamma \frac{\cos z}{z(z-1)^2}\,dz + \oint_{\gamma'} \frac{\cos z}{z(z-1)^2}\,dz$$
$$= -2\pi i(\sin 1 + \cos 1) + 2\pi i = 2\pi i(1 - \sin 1 - \cos 1).$$

Ans. $\boxed{2\pi i(1 - \sin 1 - \cos 1)}$

Problem 3 (20 points). Consider the power series

$$f(z) = \sum_{j=0}^{\infty} \frac{j+1}{j!} z^j = 1 + 2z + \frac{3}{2!} z^2 + \frac{4}{3!} z^3 + \cdots$$

(3.1) What is the radius of convergence of this power series?

Solution. Use the ratio test:

$$\lim_{n\to\infty} \frac{\frac{j+2}{(j+1)!}}{\frac{j+1}{j!}} = \lim_{n\to\infty} \frac{(j+2)j!}{(j+1)!(j+1))} = \lim_{n\to\infty} \frac{(j+2)}{(j+1)^2} = 0.$$

So $R = \infty$.

Ans. $\boxed{R = \infty}$

(3.2) Find the power series expansion for the derivative $f'(z)$.

Solution. We differentiate term-by-term:

$$f'(z) = \sum_{j=1}^{\infty} \frac{(j+1)}{j!} j z^{j-1} = \sum_{j=1}^{\infty} \frac{j+1}{(j-1)!} z^{j-1} = \sum_{n=0}^{\infty} \frac{(n+2)}{n!} z^n$$

Ans. $\boxed{\sum_{n=0}^{\infty} \frac{(n+2)}{n!} z^n}$

(3.3) Compute

$$\oint_{|z|=2} \frac{f(z)}{z^2}\,dz.$$

Solution.

$$\oint_{|z|=2} \frac{f(z)}{z^2}\,dz = 2\pi i f'(0)$$

by Cauchy's formula for the derivative. But we can read this from the power series of a function which (by Taylor's Theorem) is

$$f'(z) = f(0) + f'(0)z + \frac{f''(0)}{2!}z^2 + \cdots .$$

So looking at the given power series we see that $f'(0) = 2$, and so

$$\oint_{|z|=2} \frac{f(z)}{z^2}\,dz = 2\pi i f'(0) = 4\pi i.$$

Ans. $\boxed{4\pi i}$

(3.4) Find an explicit formula for the function $f(z)$.

Solution. We split the given power series in two parts and use the known power series expansion of e^z:

$$\begin{aligned} f(z) &= \textstyle\sum_{j=0}^{\infty} \frac{j+1}{j!} z^j = \sum_{j=0}^{\infty} \frac{j}{j!} z^j + \sum_{j=0}^{\infty} \frac{1}{j!} z^j = \sum_{j=0}^{\infty} \frac{j}{j!} z^j + e^z \\ &= z\Big(\textstyle\sum_{j=1}^{\infty} \frac{1}{(j-1)!} z^{j-1}\Big) + e^z = z\Big(\sum_{j=0}^{\infty} \frac{1}{j!} z^j\Big) + e^z = ze^z + e^z = e^z(z+1). \end{aligned}$$

Ans. $\boxed{e^z(z+1)}$

Problem 4 (20 points). Consider the function

$$f(z) = \frac{z}{(5-3z)^2},$$

which is analytic at 0 and at i.

(4.1) What is the radius of convergence of the power series of $f(z)$ in powers of z?

Solution. This function is analytic at every number except $\frac{5}{3}$ where it is not defined. So the radius of convergence of the power series about 0 is the distance from 0 to this "closest singularity", and $R = \frac{5}{3}$

Ans. $\boxed{\frac{5}{3}}$

(4.2) What is the radius of convergence of the power series of $f(z)$ in powers of $z - i$?

Solution. Same reasoning, except that now we must compute the distance from i to $\frac{5}{3}$,

$$d = \left| i - \frac{5}{3} \right| = \sqrt{1 + \frac{25}{9}} = \frac{\sqrt{34}}{3}.$$

Ans. $\boxed{\frac{\sqrt{34}}{3}}$

(4.3) Find the power series of $f(z)$ in powers of z.

Solution.

$$\frac{1}{5-3z} = \frac{1}{5}\Big(\frac{1}{1-\frac{3}{5}z}\Big) = \frac{1}{5}\sum_{j=0}^{\infty}\left(\tfrac{3}{5}\right)^j z^j$$

Differentiating this:

$$\frac{d}{dz}\Big(\frac{1}{5-3z}\Big) = \frac{1}{5}\sum_{j=1}^{\infty}\left(\tfrac{3}{5}\right)^j j z^{j-1}$$

But also

$$\frac{d}{dz}\Big(\frac{1}{5-3z}\Big) = \frac{-1}{(5-3z)^2}(-3) = \frac{3}{(5-3z)^2}$$

so that

$$\frac{3}{(5-3z)^2} = \frac{1}{5}\Big(\sum_{j=1}^{\infty}\left(\tfrac{3}{5}\right)^j j z^{j-1}\Big).$$

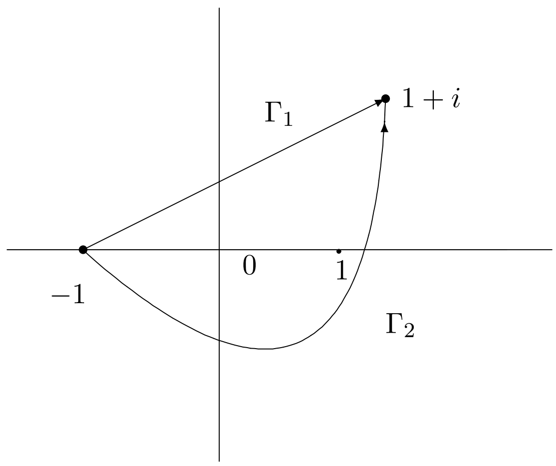
Dividing by 3 and multiplying by z we get

$$f(z) = \frac{z}{(2-3z)^2} = \frac{1}{15}\sum_{j=1}^{\infty} j\,\tfrac{3^j}{5^j} z^j = \sum_{j=1}^{\infty}\tfrac{j\,3^{(j-1)}}{5^{(j+1)}} z^j$$

(And this can be written in various ways, depending on taste.)

Ans. $\boxed{\sum_{j=1}^{\infty}\tfrac{j\,3^{(j-1)}}{5^{(j+1)}} z^j}$

Problem 5 (15 points) Consider the two contours joining -1 to $1+i$ in this figure:



(5.1) True or false (and justify your answer): $\int_{\Gamma_1} e^{-z^2} dz = \int_{\Gamma_2} e^{-z^2} dz$.

Solution. True.

Proof. The function $f(z) = e^{-z^2}$ is analytic in the entire complex plane $\mathbb{C}$; so $f(z)$ has an antiderivative $F(z)$ in $\mathbb{C}$; and so, by the Fundamental Theorem of Calculus for complex functions, for any contour Γ which joins -1 to $i+1$,

$$\int_{\Gamma} e^{-z^2} dz = F(1+i) - F(-1).$$

In particular, the values of the two integrals in the question are both equal to $F(1+i) - F(-1)$ and hence equal to each other.

(But it is another matter to compute their common value, as there is no explicit formulas for an antiderivative of e^{-z^2}.)

(5.2) Compute $\int_{\Gamma_2 - \Gamma_1} \frac{e^{-z^2}}{(z-1)} dz$.

Solution. The contour $\Gamma_2 - \Gamma_1$ is simple, closed, positively oriented and the point 1 is in its interior; so by Cauchy's formula,

$$\int_{\Gamma_2 - \Gamma_1} \frac{e^{-z^2}}{(z-1)} dz = 2\pi i e^{-z^2}\Big]_{z=1} = 2\pi i e^{-1} = \frac{2\pi i}{e}.$$

Ans. $\boxed{\frac{2\pi i}{e}}$

Problem 6 (10 points) Suppose $f(z)$ is an entire function which agrees with $\cos z$ on the circle of radius 2 around the origin, i.e.,

$$\text{if } |z| = 2, \text{ then } f(z) = \cos z.$$

You must explain how you got your answer to each of the following two questions.

(6.1) Compute $f(i)$.

Solution. Using Cauchy's formula at i,

$$f(i) = \frac{1}{2\pi i}\oint_{|w|=2} \frac{f(w)}{w-i}\,dw = \frac{1}{2\pi i}\oint_{|w|=2} \frac{\cos w}{w-i}\,dw = \cos i.$$

And we can compute this further, if we want

$$\cos i = \frac{e^{i^2} + e^{-i^2}}{2} = \frac{1}{2}(\frac{1}{e} + e).$$

Ans. $\boxed{\cos i}$

(6.2) Compute $f(3)$.

Solution. The argument in the previous part does not work , because 3 is not inside the circle $|z| = 2$. So we use Taylor's Theorem instead: since $f(z)$ is entire, it can be expanded into a power series about 0 which converges for all z,

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots = \textstyle\sum_{j=0}^{\infty} a_j z^j.$$

Moreover, each a_j can be computed from the values of $f(z)$ on any circle about 0; in fact

$$a_j = \frac{f^{(j)}}{j!}(0) = \frac{1}{2\pi i}\oint_{|z|=2} \frac{f(z)}{z^{(j+1)}}\,dz = \frac{1}{2\pi i}\oint_{|z|=2} \frac{\cos z}{z^{(j+1)}}\,dz$$

Similarly,

$$\cos z = b_0 + b_1 z + b_2 z^2 + \cdots = \textstyle\sum_{j=0}^{\infty} b_j z^j$$

where for each j,

$$b_j = \frac{1}{2\pi i}\oint_{|z|=2} \frac{\cos z}{z^{(j+1)}}\,dz = a_j.$$

It follows that $f(z)$ and $\cos z$ have the same power series expansion about 0, and so

$$f(z) = \cos z \text{ for all } z.$$

In particular, $f(3) = \cos 3$.

Note. Some students noticed in their tests that

$$\oint_{|w|=2} \frac{f(w)}{w-3}\,dw = 0$$

which is certainly true, since the function $\dfrac{f(w)}{w-3}$ is analytic on and inside the circle $|w| = 2$; but it does not help to answer the question in the problem.

Ans. $\boxed{\cos 3}$