Mathematics 132/3
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All parts of each problem are worth (approximately) the same, and there are altogether 100 points available. There is plenty of working space on the back of each page as well as a blank page at the end.

Some parts are more difficult than others: do first those parts that you can do without much thinking, and come back afterwards to work on the rest.

Good luck!

Solutions to Problems 2, 3, 6

Name: ______________________

Signature: ______________________

Problem 1. __________

Problem 2. __________

Problem 3. __________

Problem 4. __________

Problem 5. __________

Problem 6. __________

Total: ______________________

Problem 1 (50 points). Compute each of the following, and write your answers in $x+iy$ form, leaving expressions like $\cos(3), e^{\pi}$, etc. as they are. Make sure to indicate clearly whether the answer is a single number or a set of numbers, i.e., the value of a multiple-valued complex function.

(1.1) $\sqrt[3]{1-i}$

Solution.

Ans. $\square$

(1.2) $\sin\left(i+\dfrac{\pi}{2}\right)$

Solution.

Ans. $\square$

(1.3) $\log(2i)$

Solution.

Ans. $\square$

(1.4) $e^{2\log(2+3i)}$

Solution.

Ans. $\square$

(1.5) $\log(3)$

Solution.

Ans. $\square$

(1.6) $\lim_{n\to\infty}\left(\frac{1}{i+1}\right)^n$

Solution.

Ans. $\square$

(1.7) $\lim_{z\to i}\frac{z^3+1}{z(z-i)}$

Solution.

Ans. $\square$

(1.8) $f'(i)$ where $f(z) = \operatorname{Log}|z| + i \operatorname{Arg} z$

Solution.

Ans. $\square$

(1.9) $2^{i\pi}$

Solution.

Ans. $\square$

(1.10) $(2)^{\sqrt{2}}$

Solution.

Ans. $\square$

Problem 2 (10 points).

For each of the following limits, determine whether it exists or not, and if it exists, compute its value (either a complex number or ∞).

(2.1) $\lim_{n\to\infty} i^n$

Solution.

This limit does not exist, because the sequence (starting from $n = 1$) is

$$i, -1, -i, 1, i, -1, -i, 1, \ldots$$

and it just oscillates around these four points.

Ans. $\boxed{\text{The limit does not exist.}}$

(2.2) $\lim_{n\to\infty} \frac{n\sqrt[n]{2}+1}{1+3n}$

Solution.

$$\lim_{n\to\infty} \frac{n\sqrt[n]{2}+1}{1+3n} = \lim_{n\to\infty} \frac{\sqrt[n]{2}+\frac{1}{n}}{\frac{1}{n}+3} = \frac{\lim_n \sqrt[n]{2} + \lim_n \frac{1}{n}}{\lim_n \frac{1}{n} + \lim_n 3} = \frac{1}{3}.$$

Ans. $\boxed{\lim_{n\to\infty} \frac{n\sqrt[n]{2}+1}{1+3n} = \frac{1}{3}}$

(2.3) $\lim_{z\to\infty} e^{-z^2}$

Solution. $\lim_{x\to\infty} e^{-x^2} = 0$ along the real line, but $\lim_{x\to\infty} e^{-(ix)^2} = \lim_{x\to\infty} e^{x^2} = \infty$ along the "positive" imaginary line, and so $\lim_{z\to\infty} e^{-z^2}$ does not exist.

Ans. $\boxed{\text{The limit does not exist.}}$

(2.4) $\lim_{z\to 0} \cos(\frac{\sin z}{z})$

Solution.

$$\lim_{z\to 0} \cos(\frac{\sin z}{z}) = \cos\left(\lim_{z\to 0} \frac{\sin z}{z}\right) = \cos(1).$$

Ans. $\boxed{\cos(1)}$

Problem 3 (10 points)

Consider the stereographic projection, which maps the unit sphere onto the extended complex plane.

(3.1) Let C be the intersection of the unit sphere with the plane $z = \dfrac{1}{3}$ and let D be the image of C under the stereographic projection. Which of the following are true:

(1) D is a line.
(2) D is a circle which lies inside the unit circle.
(3) D is a circle which contains the unit circle.
(4) D intersects the unit circle.

Solution.

Ans. $\boxed{\text{(3), a circle which contains the unit circle.}}$

(3.2) What is the image under the stereographic projection of the circle C on the unit sphere which contains the north pole, the south pole and also the point $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)$? (Draw you answer).

Solution.

Since the north pole is on C, the image of C contains ∞, and so it is a line; and since the south pole is also on C and the projection maps the south pole onto the origin $(0, 0)$, the origin must also lie on that line: so the image is a line which goes through the origin. To decide exactly which of these lines it is, we consider also the point $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)$, which lies on the unit circle and so it is fixed by the projection: it follows that the image of this circle is the line throught the origin which contains $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, i.e., the line at an angle of $\frac{\pi}{4}$.

Ans. $\boxed{\text{The line } y = x.}$

Problem 4 (10 points) Sketch the following sets in the complex plane, and determine for each of them whether it is **open**, **closed** or a **domain**; indicate your answers by entering "YES" or "NO" in the Table below. (Some of these sets may have none or more than one of these properties.)

(a) The set consisting of the real and imaginary axes, $xy = 0$.

(b) The closed unit disk $|z| \leq 1$.

(c) The open unit disk $|z| < 1$.

(d) The open disk $|z - 2| < 1$ together with the left-half-plane $\mathrm{Re}(z) < 0$ (the union).

Answers

Set	Open?	Closed?	Domain?
(a)			
(b)			
(c)			
(d)			

Problem 5 (10 points). Consider the following two complex functions, defined in terms of their real and imaginary parts:

$$\text{(a)} \qquad f(x+iy) = (4y^2 - 4x^2 + 4x - 1) + i(8xy + 4x)$$
$$\text{(b)} \qquad g(x+iy) = (4x^2 - 4y^2 + 4y - 1) + i(8xy - 4x)$$

One of them has a derivative at $z = 0$ and the other does not.

Determine which of these two functions has a derivative at 0, and compute this derivative.

Solution.

Ans. $\square$

Problem 6 (10 points).

(6.1) Compute $\operatorname{Log}(i)$, where $\operatorname{Log}(z)$ is the principal branch of the log.

Solution.

$\operatorname{Log}(z) = \operatorname{Log}|z| + i \operatorname{Arg}(z)$, where $\operatorname{Arg}(z)$ is the principal value of the argument function, and so

$$\operatorname{Log}(i) = \frac{1}{2} i\pi.$$

Ans. $\boxed{\operatorname{Log}(i) = \frac{1}{2} i\pi.}$

(6.2) Find a branch $G(z)$ of $\log(z)$ with the following properties:

(1) The domain of $G(z)$ includes both the real and the imaginary axis (except for 0, of course).
(2) $G(i) = \operatorname{Log}(i)$.
(3) $G(1) \neq \operatorname{Log}(1)$.

Solution.

We set

$$G(z) = \operatorname{Log}|z| + i \arg_{\frac{1}{4}\pi}(z)$$

so that the branch cut of $G(z)$ is the half-line $y = x$ with $x \geq 0$, and

$$\arg_{\frac{1}{4}\pi}(i) = \frac{1}{2}\pi, \quad \arg_{\frac{1}{4}\pi}(1) = 2\pi.$$

This gives

$$G(i) = \frac{1}{2}\pi i = \operatorname{Log}(i), \qquad G(1) = 2\pi i \neq 0 = \operatorname{Log}(1).$$

Ans. $\boxed{G(z) = \operatorname{Log}|z| + i \arg_{\frac{1}{4}\pi}(z),\ G(i) = \frac{1}{2}\pi i,\ G(1) = 2\pi i.}$

(6.3) Compute $G'(i)$ for the branch you found in the previous part.

Solution. $G'(z) = \dfrac{1}{z}$ for every z in the domain of $G(z)$, since $G(z)$ is a branch of the logarithm; and so $G'(i) = \dfrac{1}{i} = -i$.

Ans. $\boxed{G'(i) = -i.}$

Math 132 Test 1 (Winter 2012)

Problem 1

(1.1) $\sqrt[3]{1-i}$

First we have $1-i = \sqrt{2}e^{-\frac{\pi}{4}i}$.
Then $\sqrt[3]{1-i} = 2^{\frac{1}{6}} e^{\frac{-\frac{\pi}{4}+2k\pi}{3}i} = 2^{\frac{1}{6}} \cos\frac{-\frac{\pi}{4}+2k\pi}{3} + i2^{\frac{1}{6}} \sin\frac{-\frac{\pi}{4}+2k\pi}{3}$
with $k = 0, 1, 2$.

(1.2) $\sin(i+\frac{\pi}{2})$

$\sin(i+\frac{\pi}{2}) = \sin i \cos\frac{\pi}{2} + \sin\frac{\pi}{2}\cos i = \cos i = \frac{e+e^{-1}}{2}$

(1.3) $\log(2i)$

$\log(2i) = \text{Log}2 + i\arg(2i) = \text{Log}2 + i(\frac{\pi}{2}+2k\pi)$ with $k \in \mathbb{Z}$

(1.4) $e^{2\log(2+3i)}$

$e^{2\log(2+3i)} = [e^{\log(2+3i)}]^2 = (2+3i)^2 = -5+12i$

(1.5) $\log(3)$

$\log(3) = \text{Log}3 + i2k\pi$ with $k \in \mathbb{Z}$

(1.6) $\lim_{n\to\infty}(\frac{1}{i+1})^n$

Since $|\frac{1}{i+1}| = \frac{1}{\sqrt{2}} < 1$, $\lim_{n\to\infty}(\frac{1}{i+1})^n = 0$.

(1.7) $\lim_{z\to i}\frac{z^3+1}{z(z-i)}$

Since $\lim_{z\to i}(z^3+1) = i^3+1 = 1-i \neq 0$, $\lim_{z\to i}\frac{z^3+1}{z(z-i)} = \infty$.

(1.8) $f'(i)$ where $f(z) = \text{Log}|z| + i\text{Arg}z$

Since $f(z) = \text{Log}|z| + i\text{Arg}z = \text{Log}z$, $f'(z) = \frac{1}{z}$ and then $f'(i) = \frac{1}{i} = -i$.

(1.9) $2^{i\pi}$

$2^{i\pi} = e^{i\pi\log 2} = e^{i\pi(\text{Log}2+i2k\pi)} = e^{-2k\pi^2+i\pi\text{Log}2} = e^{-2k\pi^2}\cos(\pi\text{Log}2) + ie^{-2k\pi^2}\sin(\pi\text{Log}2)$
with $k \in \mathbb{Z}$

(1.10) $(2)^{\sqrt{2}}$

$(2)^{\sqrt{2}} = e^{\sqrt{2}\log 2} = e^{\sqrt{2}(\text{Log}2+i2k\pi)} = e^{\sqrt{2}\text{Log}2}\cos(2\sqrt{2}k\pi)+ie^{\sqrt{2}\text{Log}2}\sin(2\sqrt{2}k\pi)$
with $k \in \mathbb{Z}$

Problem 4
(a) closed and not domain
(b) closed and not domain
(c) open and domain
(d) open but not domain

Problem 5

Consider the following two complex functions, defined in terms of their real and imaginary parts:

$$\text{(a)} \quad f(x+iy) = (4y^2 - 4x^2 + 4x - 1) + i(8xy + 4x)$$

$$\text{(b)} \quad g(x+iy) = (4x^2 - 4y^2 + 4y - 1) + i(8xy - 4x)$$

One of them has a derivative at $z = 0$ and the other does not.
Determine which of these two functions has a derivative at 0, and compute this derivative.

(a)

$$\frac{\partial(4y^2 - 4x^2 + 4x - 1)}{\partial x} = -8x + 4 \neq 8x = \frac{\partial(8xy + 4x)}{\partial y},$$

so $f(z)$ doesn't satisfy Cauchy-Riemann equations and thus nonanalytic.

(b)

$$\frac{\partial(4x^2 - 4y^2 + 4y - 1)}{\partial x} = 8x = \frac{\partial(8xy - 4x)}{\partial y}$$

and

$$\frac{\partial(4x^2 - 4y^2 + 4y - 1)}{\partial y} = -8y + 4 = -\frac{\partial(8xy - 4x)}{\partial x},$$

i.e. $g(z)$ satisfies Cauchy-Riemann equations and thus analytic everywhere.
Next

$$g'(z) = \frac{\partial(4y^2 - 4x^2 + 4x - 1)}{\partial x} + i\frac{\partial(8xy - 4x)}{\partial x} = 8x + i(8y - 4),$$

then $g'(0) = -4i$.