Mathematics 132/1
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All parts of each problem are worth (approximately) the same, and there are 108 points altogether as indicated. There is plenty of working space on the back of each page as well as a blank page at the end.

Good luck!

Solved test

Name: ______________________

Signature: ______________________

Problem 1. __________

Problem 2. __________

Problem 3. __________

Problem 4. __________

Total: ______________________

Problem 1 (10 points). Multiple choice questions about the stereographic projection. You do not need to justify your answers, but you will be penalized for wrong choices.

(1.1) Let C be the intersection of the unit sphere S with the plane $x + y + z = 1$. The image of C under the stereographic projection is

(a) A line through the origin.
(b) A line which does not pass through the origin.
(c) A circle.

Solution. The plane $x + y + z = 1$ goes through the North Pole, and so the image of this circle is a line; but it does not go through the South Pole, and so the origin is not on that line.

Ans. $\boxed{\text{b}}$

(1.2) Let C be the intersection of the unit sphere S with the plane $x = \frac{1}{2}$. The image of C under the stereographic projection is

(a) A line through the origin.
(b) A line which does not pass through the origin.
(c) A circle.

Solution. This plane does not go through the North Pole, and so the image of the circle is a circle.

Ans. $\boxed{\text{c}}$

(1.3) Let C be the intersection of the unit sphere S with the plane $z = -\frac{1}{2}$. The image of C under the stereographic projection is

(a) A line through the origin.
(b) A line which does not pass through the origin.
(c) A circle.

Solution. This plane does not go through the North Pole, and so the image of the circle is a circle.

Ans. $\boxed{\text{c}}$

Problem 2 (63 points). Compute each of the following, and write your answers in $x+iy$ form, leaving expressions like $\cos(3), e^{\pi}$, etc. exact. Make sure to indicate clearly whether the answer is a single number or a set of numbers, i.e., the value of a multiple-valued complex function.

(2.1) $e^{\frac{1}{1+i}}$

Solution.

$$\frac{1}{1+i}=\frac{1-i}{(1+i)(1-i)}=\frac{1-i}{2}=\frac{1}{2}-\frac{i}{2};$$

and so

$$e^{\frac{1}{1+i}}=e^{\frac{1}{2}-\frac{1}{2}i}=\sqrt{e}(\cos(-\frac{1}{2})+i\sin(-\frac{1}{2}))=\sqrt{e}(\cos(\frac{1}{2})-i\sin(\frac{1}{2}))$$

Ans. $\boxed{\sqrt{e}(\cos(\tfrac{1}{2})-i\sin(\tfrac{1}{2}))\text{ (one value)}}$

(2.2) $\log\left(\frac{1}{1+i}\right)$

Solution. We use the fact

$$\frac{1}{1+i}=\frac{1}{2}-\frac{i}{2}$$

from the previous problem, to get

$$\begin{aligned}\log(\frac{1}{1+i})&=\log(\frac{1}{2}-\frac{i}{2})=\operatorname{Log}\left|\frac{1}{2}-\frac{i}{2}\right|+i\arg\left(\frac{1}{2}-\frac{i}{2}\right)\\&=\operatorname{Log}(\frac{1}{2})+i\frac{-\pi}{4}+2k\pi i=-\operatorname{Log}(2)+\frac{(8k-1)i\pi}{4},k=0,\pm1,\pm2,\ldots\end{aligned}$$

Ans. $\boxed{-\operatorname{Log}(2)+\tfrac{(8k-1)i\pi}{4},k=0,\pm1,\pm2,\ldots\text{ (infinitely many values)}}$

(2.3) $(1+i)^{48}$

Solution.

$$(1+i)^{48}=\left(\sqrt{2}e^{\frac{i\pi}{4}}\right)^{48}=\sqrt{2}^{48}e^{\frac{48i\pi}{4}}=2^{24}e^{12i\pi}=2^{24}.$$

Ans. $\boxed{2^{24}\text{ (one value)}}$

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Problem 2 continued from the previous page

(2.4) $\sqrt[6]{-1}$

Solution.

$$\sqrt[6]{-1} = \left(e^{i(\pi+2k\pi)}\right)^{\frac{1}{6}} = e^{i\pi\frac{2k+1}{6}} \quad k = 0, \pm 1, \pm 2, \ldots$$

This expression gives six distinct values for $k = 0$ - 5, before it starts repeating; the values are

$$e^{i\frac{\pi}{6}},\ e^{i\frac{\pi}{2}},\ e^{i\frac{5\pi}{6}},\ e^{i\frac{7\pi}{6}},\ e^{i\frac{3\pi}{2}},\ e^{i\frac{11\pi}{6}}.$$

These numbers can also be easily written in $x + iy$ form: for example,

$$e^{i\frac{\pi}{6}} = \frac{\sqrt{3}}{2} + \frac{i}{2}.$$

Ans. $\boxed{e^{i\frac{\pi}{6}},\ e^{i\frac{\pi}{2}},\ e^{i\frac{5\pi}{6}},\ e^{i\frac{7\pi}{6}},\ e^{i\frac{3\pi}{2}},\ e^{i\frac{11\pi}{6}}\ \text{(six values)}}$

(2.5) $(-2)^5$

Solution. This is an integer power, and it is computed by multiplying -2 by itself five times; so $(-2)^5 = -32$.

Ans. $\boxed{-32\ \text{(one value)}}$

(2.6) 2^π

Solution.

$$2^\pi = e^{\pi \log(2)} = e^{\pi[\mathrm{Log}(2)+i\arg(2)]} = e^{\pi[\mathrm{Log}(2)+i2k\pi]} = e^{\pi\,\mathrm{Log}(2)}e^{\pi 2k\pi i},$$
$$= e^{\pi\,\mathrm{Log}(2)}e^{2k\pi^2 i} = \quad k = 0, \pm 1, \pm 2, \ldots .$$

Ans. $\boxed{e^{\pi\,\mathrm{Log}(2)}e^{2k\pi^2 i} = \quad k = 0, \pm 1, \pm 2, \ldots\ \text{(infinitely many vales)}}$

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Problem 2 continued from the previous page

(2.7) $(-2)^\pi$

Solution.

$$(-2)^\pi = e^{\pi \log(-2)} = e^{\pi[\text{Log}(2)+i\arg(-2)]} = e^{\pi[\text{Log}(2)+i\pi+i2k\pi]}$$
$$= e^{\pi\,\text{Log}(2)}e^{(2k+1)\pi^2 i} = \qquad k = 0, \pm1, \pm2, \ldots .$$

Ans. $\boxed{e^{\pi\,\text{Log}(2)}e^{(2k+1)\pi^2 i} = \qquad k = 0, \pm1, \pm2, \ldots \text{ (infinitely many values)}}$

(2.8) 1^i

Solution.

$$1^i = e^{i\log(1)} = e^{i[\text{Log}\,1+2k\pi i]} = e^{2k\pi i^2} = e^{-2k\pi}, \quad k = 0, \pm1, \pm2, \ldots .$$

Notice that this can also be written as

$$e^{2k\pi}, \quad k = 0, \pm1, \pm2, \ldots ,$$

because the $-2k$ stands for an arbitrary (positive or negative) even integer.

Ans. $\boxed{e^{2k\pi}, \quad k = 0, \pm1, \pm2, \ldots \text{ (infinitely many, real values)}}$

(2.9) $(-1)^i$

Solution.

$$(-1)^i = e^{i\log(-1)} = e^{i[\text{Log}\,1+\pi i+2k\pi i]} = e^{(2k+1)\pi i^2} = e^{-(2k+1)\pi}, \quad k = 0, \pm1, \pm2, \ldots .$$

Notice that this can also be written as

$$e^{(2k+1)\pi}, \quad k = 0, \pm1, \pm2, \ldots ,$$

because the $-(2k+1)$ stands for an arbitrary (positive or negative) odd integer.

Ans. $\boxed{e^{(2k+1)\pi}, \quad k = 0, \pm1, \pm2, \ldots , \text{ (infinitely many, real values)}}$

Problem 3 (15 points). Compute the following limits and derivatives:

(3.1) $\lim_{n\to\infty} \frac{(i+\frac{1}{n})^2+1}{n}$

Solution.

$$\lim_{n\to\infty} \frac{(i+\frac{1}{n})^2+1}{n} = \lim_{n\to\infty} \frac{i^2+2i\frac{1}{n}+\frac{1}{n^2}+1}{n} = \lim_{n\to\infty} \frac{2i\frac{1}{n}+\frac{1}{n^2}}{n}$$
$$= \lim_{n\to\infty}(\frac{2i}{n^2}+\frac{1}{n^3}) = 0.$$

Ans. $\boxed{0}$

(3.2) $\lim_{z\to i} \frac{z^3+i}{z-i}$

Solution. The numerator of this fraction factors:

$$z^3+i = z^3-(i^3) = (z-i)(z^2+zi+i^2) = (z-i)(z^2+zi-1);$$

so

$$\lim_{z\to i} \frac{z^3+i}{z-i} = \lim_{z\to i} \frac{(z-i)(z^2+zi-1)}{z-i} = \lim_{z\to i}(z^2+zi-1) = i^2+ii-1 = -3.$$

Ans. $\boxed{-3}$

(3.3) $\frac{d}{dz}(z^2+i)^{17}$

Solution. By the Chain Rule, directly:

$$\frac{d}{dz}(z^2+i)^{17} = 17(z^2+1)^{16}(2z) = 34z(z^2+1)^{16}.$$

Ans. $\boxed{34z(z^2+1)^{16}}$

Problem 4 (20 points). This is about branches.

(4.1) Find a branch $F(z)$ of the multiple valued function $\sqrt[3]{z}$ which is continuous on the left-half-plane $\mathrm{Re}(z) < 0$, and compute $F(i)$.

Solution. By definition, the multiple-valued cube root function is given by

$$\sqrt[3]{z} = e^{\frac{1}{3}\log z}$$

and every branch of the logarithm defines a branch of it. We need a branch which is continuous on the left-half-plane and also defined at i, and the simplest choice is

$$F(z) = e^{\frac{1}{3}\log_0(z)} = e^{\frac{1}{3}[\mathrm{Log}\,|z| + i\arg_0(z)]},$$

where

$$\arg_0(z) = \text{ the unique } \theta \text{ such that } z = |z|e^{i\theta} \text{ and } 0 < \theta < 2\pi.$$

The branch cut for $\log_0(z)$ is the positive real axis $[0, \infty)$, and so it is continuous at every z which is not real and positive. Also, $\arg_0(i) = \frac{\pi}{2}$, and so

$$F(i) = e^{\frac{1}{3}[\mathrm{Log}\,|i| + i\frac{\pi}{2}]} = e^{\frac{\pi}{6}} = \frac{\sqrt{3}}{2} + i\frac{1}{2}.$$

Ans. $\boxed{F(z) = e^{\frac{1}{3}[\mathrm{Log}\,|z| + i\arg_0(z)]},\ F(i) = \frac{\sqrt{3}}{2} + \frac{i}{2}.}$

(4.2) Find a branch $G(z)$ of the multiple valued function $\log(iz)$ which is continuous on the upper-half-plane $\mathrm{Im}(z) > 0$ and compute $G(i-1)$.

Solution. When z is in the upper-half-plane, iz is the left-half-plane, and so to get a branch of $\log(iz)$ which is continuous on the upper-half-plane, we choose again $\log_0(w)$, which is continuous on the left-half-plane:

$$G(z) = \log_0(iz) = \mathrm{Log}\,|iz| + i\arg_0(iz).$$

For the required value, we compute:

$$G(i-1) = \log_0(i(i-1)) = \log_0(-1-i) = \mathrm{Log}\,|-1-i| + i\arg_0(-1-i)$$
$$= \mathrm{Log}(\sqrt{2}) + \frac{5\pi i}{4}.$$

Notice that we could choose any branch of the logarithm here whose branch cut is not in the left-half-plane, e.g., $\log_{\frac{\pi}{2}}(w)$ or $\log_{2\pi}(w)$, etc. These branches are all correct solutions, and some of them yield different values for $G(i-1)$; for example, if we set

$$G_1(z) = \log_{2\pi}(iz) = \mathrm{Log}\,|iz| + i\arg_{2\pi}(iz),$$

then $G_1(i-1) = \mathrm{Log}(\sqrt{2}) + \frac{13\pi i}{4}$.

Ans. $\boxed{G(z) = \log_0(iz) = \mathrm{Log}\,|iz| + i\arg_0(iz),\ G(i-1) = \mathrm{Log}(\sqrt{2}) + \frac{5\pi i}{4}.}$
