Mathematics 132/1
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There are 220 points available on this test, as indicated, and there is plenty of space on the backs of pages and on the blank page at the end.

Be sure to indicate clearly which is your final answer to each problem, and put it in the box provided where this is appropriate. You should give numerical answers in the form $x + iy$, but you don't need to approximate the real numbers x and y. So $3 - \pi i$ and $\cos 1 - i \sin 1$ are OK, but $\cos i$ or e^i are not.

All simple closed contours are oriented positively (counterclockwise).

Good luck!

Solved test

Name: ____________________

Signature: ____________________

Problem 1. __________

Problem 2. __________

Problem 3. __________

Problem 4. __________

Problem 5. __________

Problem 6. __________

Total: ____________________

Problem 1 (100 points). Show your work, unless the answer is completely obvious, and indicate clearly whether there is a single answer or a set of answers—the value of a multiple-valued function.

(1.1) Compute $\dfrac{e^i}{1-i}$.

Ans. $\boxed{\text{One value, } \frac{1}{2}[(\cos 1 - \sin 1) + i(\cos 1 + \sin 1)]}$

(1.2) Compute $\log(1 + \sqrt{3}i)$.

Ans. $\boxed{\text{Inf. many values, } \operatorname{Log} 2 + i(\frac{\pi}{3} + 2k\pi)}$

(1.3) Compute $(1 + \sqrt{3}i)^{107}$.

Ans. $\boxed{\text{One value, } 2^{106}(1 - \sqrt{3})}$

(Problem 1 continues on the next page.)

(Continuing with Problem 1.)

(1.4) Compute $3^{(1+\sqrt{3}i)}$

Ans. $\boxed{\text{Inf. many values, } 3^{-2k\pi\sqrt{3}}(\cos(\sqrt{3}\,\mathrm{Log}\,3) + i\sin(\sqrt{3}\,\mathrm{Log}\,3))}$

(1.5) Compute $\displaystyle\oint_{|z|=1} \frac{dz}{(2z-i)^5}$.

Ans. $\boxed{\text{One value, } 0}$

(Problem 1 continues on the next page.)

(1.6) Suppose $f(z)$ has a pole of order 3 at 0. Compute $\lim_{z\to 0} f(z)\sin(z)$. ($\infty$ or "does not exist" are allowed as answers.)

Ans. $\boxed{\infty}$

(1.7) Suppose $f(z)$ has a pole of order 3 at 0. Compute $\lim_{z\to 0} f(z)\sin(z^4)$. ($\infty$ or "does not exist" are allowed as answers.)

Ans. $\boxed{0}$

(Problem 1 continues on the next page.)

(Continuing with Problem 1.)

(1.8) Find the radius of convergence of $\sum_{k=2}^{\infty} k^2 z^k$.

Ans. $\boxed{1}$

(1.9) Compute $\oint_{|z|=2} z^3 \cos\left(\frac{1}{z^2}\right) dz$.

Ans. $\boxed{-\pi i}$

(1.10) Classify the singularity of $\frac{e^{\frac{1}{z}}}{z^2 - 1}$ at 1 and compute the residue.

Ans. $\boxed{\text{Pole of order 1, residue } = \frac{e}{2}}$

Problem 2 (20 points).

(2.1) Prove that $u = e^{-y} \cos x$ is harmonic in the entire plane and find a conjugate of it.

Ans. $\boxed{v = e^{-y} \sin x}$

(Problem 2 continues on the next page.)

(Continuing with Problem 2.)

(2.2) Suppose $f(z)$ is analytic and never 0 in a domain D, and set

$$u = \operatorname{Log}|f(z)| \qquad \text{(the real logarithm).}$$

Prove that u is harmonic in D.

Hint. Do not compute; to prove that u satisfies Laplace's equation at each (x, y) use an appropriately chosen branch of $\log w$.

Solution. For any z_0 in the domain D, let $w_0 = f(z_0)$. This is not 0 by the hypothesis, and so there is a branch $\log_\tau(w)$ of the logarithm whose branch cut avoids it. This means that the function

$$F(z) = \log_\tau(f(z)) = \operatorname{Log}|f(z)| + i \arg_\tau(f(z))$$

is analytic "near z_0" (in some disc about z_0), and so its real part $\operatorname{Log}|f(z)|$ is harmonic in that disc.

Problem 3 (30 points). Find the Laurent expansion about 0 of the function

$$f(z) = \Big(\frac{z}{3z-1}\Big)^2 + \frac{1}{(z-1)^2}$$

in the annulus $\frac{1}{3} < |z| < 1$. Make a drawing.

Ans. $\boxed{\sum_{n=0}^{\infty} \Big[\frac{(n+1)}{3^{n+2}}\frac{1}{z^n} + (n+1)z^n\Big]}$

Problem 4 (20 points). Let C be the positively oriented circle $|z| = \dfrac{1}{2}$.

(4.1) Write the value of the following integral as the sum of an infinite series,

$$\oint_C \frac{e^{\frac{1}{z}}}{z(z-1)^2}\,dz = a_0 + a_1 + a_2 + \cdots = \textstyle\sum_{n=0}^{\infty} a_n$$

Solution.

$$e^{\frac{1}{z}} = \sum_{k=0}^{\infty} \frac{1}{k!z^k},$$

$$\frac{1}{z-1} = \frac{-1}{1-z} = -\sum_{k=0} z^k,$$

$$\frac{-1}{(z-1)^2} = -\frac{d}{dz}\Big(\sum_{k=0} z^k\Big) = -\sum_{k=1}^{\infty} kz^{k-1} = -\sum_{k=0}^{\infty}(k+1)z^k$$

and hence

$$\frac{e^{\frac{1}{z}}}{z(z-1)^2} = \frac{1}{z}\Big(\sum_{k=0}^{\infty}\frac{1}{k!z^k}\Big)\Big(\sum_{k=0}^{\infty}(k+1)z^k\Big).$$

We only need the constant term of the product of these two infinite series, because that is the coefficient of $\dfrac{1}{z}$ in the Laurent series of $\dfrac{e^{\frac{1}{z}}}{z(z-1)^2}$, and this is

$$\mathrm{Res}\Big(\frac{e^{\frac{1}{z}}}{z(z-1)^2}\Big) = \sum_{k=0}^{\infty}\frac{k+1}{k!}.$$

Ans. $\boxed{2\pi i \sum_{n=0}^{\infty} \frac{(n+1)}{n!}}$

(4.2) Compute this sum $\sum_{n=0}^{\infty} a_n$ in Part 4.1 to get a numeric value for the integral.

Solution.

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!},$$

$$ze^z = \sum_{k=0}^{\infty} \frac{z^{k+1}}{k!},$$

$$\frac{d}{dz}(ze^z) = \sum_{k=0}^{\infty} \frac{(k+1)z^k}{k!},$$

where we do not skip the term for $k = 0$ because the series has no constant term, is starts with $\frac{z}{0!} = z$ whose derivative is 1. It follows that

$$e^z + ze^z = \sum_{k=0}^{\infty} \frac{(k+1)z^k}{k!}.$$

If we plug $z = 1$ in this equation, we get

$$2e = \sum_{k=0}^{\infty} \frac{(k+1)}{k!}$$

and so the sum of the series in the first part is $2e$.

Ans. $\boxed{\text{The series sums up to } 2e\text{, and so the integral is } 4\pi i e}$

Problem 5 (30 points).

(5.1) Compute $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}$.

Ans. $\boxed{\frac{\pi}{2}}$

(Problem 5 continues on the next page.)

(Continuing with Problem 5.)

(5.2) Compute $\int_{-\infty}^{\infty} \frac{\cos x \, dx}{(1+x^2)^2}$.

Ans. $\boxed{\frac{\pi}{e}}$

Problem 6 (20 points).

(6.1) Suppose that $f(z)$ is an entire function such that for some sequence $z_0, z_1, \ldots$ with $z_n \neq 0$,

$$\lim_{n\to\infty} z_n = 0 \text{ and for every } n,\ f(z_n) = 0.$$

Prove that for every z, $f(z) = 0$. (*Hint.* Start with the fact that $f(0) = 0$.)

Solution. Since $f(z)$ is entire,

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$

and $a_0 = f(0) = \lim_{n\to\infty} f(z_n) = 0$. Suppose towards a contradiction that $f(z)$ is not identically 0, so there is a least m such that $a_m \neq 0$, and

$$f(z) = a_m z^m + a_{m+1} z^{m+1} + \cdots = z^m h(z)$$

where $h(z) = a_m + a_{m+1} z + \cdots$ is analytic and $h(0) = a_m \neq 0$. But

$$\lim_{n\to\infty} h(z_n) = \lim_{n\to\infty} \frac{f(z_n)}{(z_n)^m} = 0,$$

which is the desired contradiction.

(6.2) True or false: *If $f(z)$ is an entire function which is bounded on the left-half-plane* $\mathrm{Re}(z) \leq 0$, *then $f(z)$ is constant in the left-half-plane.* You must prove it if you choose "true" or provide a counterexample if you choose "false".

Solution. This is false: counterexample is e^z for which if $x \leq 0$,

$$|e^z| = |e^{x+iy}| = |e^x||e^{iy}| = |e^x| \leq 1$$

since for $x \leq 0$, $e^x \leq 1$.
