Mathematics 132/3
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Final Examination, Thursday, March 17, 2009

There are 220 points in the test, with 200 counting as a perfect score, and there is ample time for you to work on all the problems; don't rush, and check your answers whenever this is possible.

Good luck!

Note: All closed contours are positively oriented.

Name: ____________________

Signature: ____________________

Problem 1. __________

Problem 2. __________

Problem 3. __________

Problem 4. __________

Problem 5. __________

Problem 6. __________

Problem 7. __________

Problem 8. __________

Problem 9. __________

Total: ____________________

Problem 1 (15 pts)**.** Compute each of the following expressions in the form $x + iy$, indicating clearly whether the answer is a single number or a set of numbers, i.e., the value of a multiple-valued function.

(1a). $\sqrt[6]{-1}$

Ans.

(1b). $\log(1 + i)$

Ans.

(1c). $\cos(i\pi)$

Ans.

Problem 2 (20 pts)**.** Suppose $f(z)$ is analytic at 0, and

$$f(0) = 1, \quad f^{(n)}(0) = n3^n.$$

(2a). Find the power series expansion of $f(z)$ in powers of z.

Ans. $\boxed{}$

(2b). Compute the radius of convergence of this power series.

Ans. $\boxed{}$

Problem 3 (25 pts)**.** Find the Laurent series (in powers of z), for the function

$$f(z) = \frac{1}{3z-1} + \frac{z}{(z-2)^2}$$

in the annulus $\frac{1}{3} < |z| < 2$.

Ans. $\boxed{}$

Problem 4 (20 pts)**.** For each of the following functions, classify the singularity at 0 and find the residue.

(4a). $\dfrac{\sin z}{z^2}$.

Ans. $\boxed{}$

(4b). $\dfrac{\sin z}{z^3} + \dfrac{\sin z}{z^2}$.

Ans. $\boxed{}$

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(4c). $\dfrac{\cos z - 1}{z^5}$.

Ans.

(4d). $\dfrac{\operatorname{Log}(1-z)}{z^3}$, where $\operatorname{Log}(z)$ is the principal branch of the logarithm.

Ans.

Problem 5 (45 pts)**.** Compute each of the following four contour integrals (with the contours positively oriented).

Be careful: each part requires some thinking, and one of the parts is substantially more complex than the others! You are not required to prove your answer, but your work should show clearly how you computed it. (Use the last, blank page if you need scratch paper.)

(5a). $\displaystyle\oint_{|z|=1} \frac{e^z dz}{z-2}$.

Ans.

(5b). $\displaystyle\oint_{|z|=3} \frac{e^z dz}{z-2}$.

Ans.

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(5c). $\oint_{|z|=1} z^3 e^{\frac{1}{z}} dz.$

Ans. $\boxed{}$

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(5d). $\displaystyle\oint_{|z|=1} \frac{e^{\frac{1}{z}}\,dz}{z-2}$.

Ans. $\boxed{}$

Problem 6 (15 pts)**.** Compute the integral $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)}$

Ans. $\boxed{}$

Problem 7 (20 pts)**.** Two problems on fractional linear transformations.

(7a). Find a fractional linear transformation which takes the left halfplane $\mathrm{Re}(z) < 0$ onto the open disk $|z - 1| < 1$. (Make a drawing.)

Ans.

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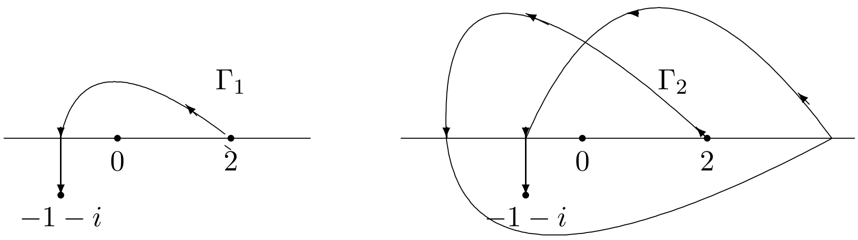
(7b). Find a fractional linear transformation which takes the open unit disk $|z| < 1$ onto itself and takes 0 to $\frac{1}{2}$. (Make a drawing.)

Ans.

Problem 8 (30 pts)**.** The two parts of this problem ar about the two parts of this drawing:



(8a). For the contour Γ_1 in the drawing which starts at 2 and ends at $-1-i$, compute

$$\int_{\Gamma_1} \frac{dz}{z}$$

(You don't need to prove your answer, but you must show the work which justifies it.)

Ans.

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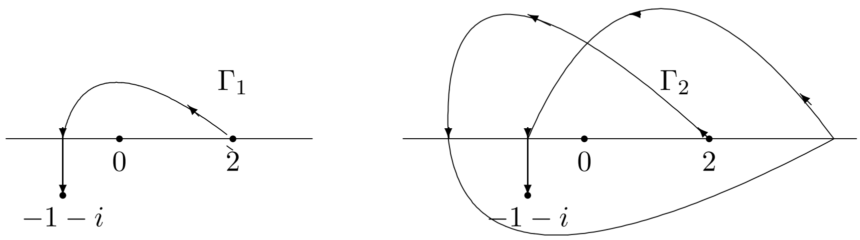

(8b). For the contour Γ_2 in the drawing which starts at 2 and ends at $-1-i$ (after it crosses itself), compute

$$\int_{\Gamma_2} \frac{dz}{z}$$

(You can use your answer to part (8a) to do this easily, by breaking the integral into **two** parts.)

Ans.

Problem 9 (30 pts)**.** Consider the multiple valued function

$$f(z) = \log\left(\frac{1-z}{1+z}\right),$$

defined when $z \neq 1, z \neq -1$.

(9a). Find a branch $F_1(z)$ of $f(z)$ which is analytic in the open unit disk $|z| < 1$, and compute $F_1(\frac{1}{2})$. (Make a drawing.)

Ans. $\boxed{}$

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(9b). Find a branch $F_2(z)$ of $f(z)$ which is analytic in the exterior of the open unit disk, i.e., $|z| > 1$, and compute $F_2(1 + 2i)$. (Make a drawing.)

Ans.