$18 + 12 + 20 + 20 + 15$ 85

Math 132 Midterm
May 9, 2014

Name: ______________________________

Student ID: ______________________________

Instructions: Show all of your work, and clearly indicate your answers. Use the backs of pages as scratch paper. No books, other paper, or calculators are allowed.

1. (20 points) Express the following complex numbers in either polar form or standard form:

   (a). $i^{\frac{1}{5}}$.
   (b). $\mathrm{Log}(1+i)$.
   (c). i^i.
   (d). 2^i.

a) $i^{1/5} = e^{(\frac{\pi}{2}+2\pi k)i(\frac{1}{5})} = e^{(\frac{\pi}{10}+\frac{2\pi k}{5})i} = \cos\left(\frac{\pi}{10}+\frac{2\pi k}{5}\right) + i\sin\left(\frac{\pi}{10}+\frac{2\pi k}{5}\right)$ where $k = \pm 0, 1, 2$

b) $\mathrm{Log}(1+i) = \log|1+i| + i\,\mathrm{Arg}(1+i) + 2\pi i k = \log\sqrt{2} + \frac{\pi}{4}i + 2\pi i k$

$= \log\sqrt{2} + i\left(\frac{\pi}{4} + \cancel{2\pi k}\right)$

principle value.

$-2.$

c) $i^i = e^{(\frac{\pi}{2}+2\pi k)i \cdot i} = e^{-\frac{\pi}{2}-2\pi k}$

d) $2^i = e^{\ln 2^i} = e^{i\ln 2} = e^{i(\ln 2 + 0 + 2\pi k i)} = e^{i\ln 2 - 2\pi k}$

$= e^{-2\pi k}e^{i\ln 2}$

$= e^{-2\pi k}(\cos(\ln 2) + i\sin(\ln 2))$

$e^{\ln i^i}$

$e^{i\ln i} = e^{i(\ln 1 + i\frac{\pi}{2} + 2\pi k i)}$

$e^{\ln i^{1/5}} \quad e^{\frac{1}{5}\ln i} = e^{\frac{1}{5}(0 + \frac{\pi}{2}i + 2\pi k i)}$

2. (20 points) Prove the triangle inequality

$$|z+w| \leq |z| + |w|$$

for any two complex numbers z and w. When the equality holds?

$$|z+w|^2 = (z+w)(\overline{z+w}) = (z+w)(\bar{z}+\bar{w}) = z\bar{z} + z\bar{w} + \bar{z}w + w\bar{w}$$

$$= |z|^2 + z\bar{w} + \bar{z}w + |w|^2$$

$$= |z|^2 + z\bar{w} + \overline{z\bar{w}} + |w|^2 \quad , \quad \text{note } z\bar{w} + \overline{z\bar{w}} = 2\,\mathrm{Re}\{z\bar{w}\}$$

$$= |z|^2 + 2\,\mathrm{Re}\{z\bar{w}\} + |w|^2$$

$$|z\bar{w}| \geq \mathrm{Re}\{z\bar{w}\}$$

since

$|z| = \sqrt{x^2+y^2}$

where as

$\mathrm{Re}\{z\} = x$

so $$|z+w|^2 \leq |z|^2 + 2|z\bar{w}| + |w|^2 = |z|^2 + 2|z||w| + |w|^2$$

$$|z+w|^2 \leq (|z|+|w|)^2$$

$$|z+w| \leq |z| + |w|$$

This ~~eq~~ inequality holds when $z, w \in \mathbb{C}$

$|z+w| = |z| + |w|$

when z & w are ~~positive~~ real numbers

is true. but ....

3. (20 points) If $f = u + iv$ is analytic in a domain D, prove that ∇u is perpendicular to ∇v at any point in D.

$f = u + iv$ is analytic

by Cauchy Riemann equations

$U_x = V_y \qquad \frac{dU}{dx} = \frac{dv}{dy}$

$U_y = -V_x \qquad \frac{du}{dy} = -\frac{dv}{dx}$

$\nabla U = \frac{dU}{dx}\hat{x} + \frac{dU}{dy}\hat{y}$

$\nabla V = \frac{dV}{dx}\hat{x} + \frac{dV}{dy}\hat{y} = \frac{-dU}{dy}\hat{x} + \frac{dU}{dx}\hat{y}$

when two vectors are $\perp$, their dot product is zero

$\nabla U \cdot \nabla V = \frac{dU}{dx} \cdot \frac{-dU}{dy} + \frac{dU}{dy} \cdot \frac{dU}{dx} = 0$

Therefore, for all f analytic in D, C-R equations hold and $\nabla U \perp \nabla V$ for any point in D.

4. (20 points) Show that $xy + 3x^2y - y^3$ is harmonic and find its harmonic conjugate.

$u = xy + 3x^2y - y^3$

$u_x = y + 6xy$ $\quad\quad u_y = x + 3x^2 - 3y^2$

$u_{xx} = 6y$ $\quad\quad u_{yy} = -6y$

$u_{xx} + u_{yy} = 0$ so $xy + 3x^2y - y^3$ is harmonic.

Now find harmonic conjugate.

$u_x = v_y$

$y + 6xy = u_x$

$v_y = y + 6xy$

$v = \frac{1}{2}y^2 + 3xy^2 + h(x)$

$v_x = 3y^2 + h'(x)$

$v_x = -u_y = -(x + 3x^2 - 3y^2) = 3y^2 - x - 3x^2$

$3y^2 + h'(x) = 3y^2 - 3x^2 - x$

$h'(x) = -3x^2 - x$

$h(x) = -x^3 - \frac{1}{2}x^2 + C$

So $v = \frac{1}{2}y^2 + 3xy^2 - x^3 - \frac{x^2}{2} + C$

is harmonic conjugate.

$v_x = 3y^2 - 3x^2 - x$

5. (20 points) Given any z_0 in the complex plane. Evaluate the integral

$$\oint_{|z-z_0|=2} \frac{e^z}{(z-z_0)^m} dz$$

where m is an integer.

By cauchy-integral formula

$$\frac{m!}{2\pi i} \oint_{\partial D} \frac{f(w)}{(w-z)^{m+1}} dw = f^{m}(z)$$

$$\oint_{|z-z_0|=2} \frac{e^z}{(z-z_0)^m} dz = f^{m-1}(z_0) \frac{2\pi i}{(m-1)!}$$

$$\oint_{|z-z_0|=2} \frac{e^z}{(z-z_0)^m} = e^{z_0} \left(\frac{2\pi i}{(m-1)!} \right)$$

since $f^{m-1}(z) = e^z$

e^z analytic in $|z-z_0|=2$

$\frac{e^z}{(z-z_0)^m}$ is not analytic in $|z-z_0|=1$

so Cauchy theorem $\int_{\partial D} f(z)\,dz = 0$ does not hold here

so $$\oint_{|z-z_0|=2} \frac{e^z}{(z-z_0)^m} = \frac{2\pi i e^{z_0}}{(m-1)!}$$

when $m < 0$?