Math 132, Lecture 2
Midterm

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- **There are five questions on this examination.**
- **You may not consult any books, notes, students, calculators, electronic devices, or any other aids during the exam.**
- **You have 50 minutes to complete this exam.**
- **Show your work.**

QUESTION	VALUE	SCORE
1	10	10
2	10	10
3	10	10
4	10	8
5	10	10
TOTAL	50	48

1. (10 points) Show that

$$|2z - i| < |2 + iz|$$

for all complex numbers z with $|z| < 1$.

10

$$|2(x+iy) - i|^2 = |2x + 2iy - i| = |2x + i(2y-1)|^2$$

$$= 4x^2 + (2y-1)^2 = 4x^2 + 4y^2 - 4y + 1$$

$$|2 + i(x+iy)|^2 = |2 + ix - y|^2 = |(2-y) + ix|^2$$

$$= (2-y)^2 + x^2 = 4 - 4y + y^2 + x^2$$

$$|2z - i| < |2 + iz|$$

$$\Rightarrow \quad 4x^2 + 4y^2 - \cancel{4y} + 1 < 4 - \cancel{4y} + y^2 + x^2$$

$$\Rightarrow \quad 3x^2 + 3y^2 < 3$$

$$\Rightarrow \quad x^2 + y^2 < 1$$

$$\Rightarrow \quad |z|^2 < 1$$

$$\Rightarrow \quad |z| < 1$$

$\therefore$ $|2z - i| < |2 + iz|$ only when $|z| < 1$ $\forall z \in \mathbb{C}$ ∎

2. (10 points) A *branch of* $z^{1/3}$ on a domain D is a function $f(z)$ such that $f^3 = z$ for all z in D. In what follows, let D be the upper half-plane $\{\text{Im}(z) > 0\}$.

1a

a) **(4 pts.)** Recalling that the principal branch of the logarithm $\text{Ln}(z)$ is defined by:

$$\text{Ln}(z) = \ln|z| + i\ \text{Arg}(z),$$

show that

$$f(z) = e^{\text{Ln}(z)/3}$$

defines a branch of $z^{1/3}$.

1. $z^{1/3} = e^{1/3 \ln(z)} = e^{1/3(\ln|z| + i\,\text{Arg}(z) + i2\pi n)} = f_n(z)$

2. One branch of $z^{1/3}$ is defined by $f_0(z)$.

$$f_0(z) = e^{1/3(\ln|z| + i\,\text{Arg}(z) + 0)}$$

$$= e^{1/3(\ln|z| + i\,\text{Arg}(z))}$$

$$= e^{\frac{\text{Ln}(z)}{3}}$$

$\therefore\ f(z) = e^{\frac{\text{Ln}(z)}{3}}$ defines a branch of $z^{1/3}$

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b) **(3 pts.)** Use $\text{Ln}(z)$ to find a formula for the branch f_1 of $z^{1/3}$ on D for which $f_1(i) = e^{5\pi i/6}$.

1. $f_1(i) = e^{\frac{1}{3}(\ln|i| + i\,\text{Arg}(i) + i2\pi n)} = e^{\frac{1}{3}(i\frac{\pi}{2} + i2\pi n)}$

Want $\cancel{i}\frac{\pi}{3}(\frac{1}{2} + 2n) = i\frac{5\pi}{6_2} \Rightarrow \frac{1}{2} + 2n = \frac{5}{2} \Rightarrow 2n = 2 \Rightarrow n = 1$

2. $f_1 = e^{\frac{1}{3}(\ln|z| + i\,\text{Arg}\,z + i2\pi(1))} = e^{\frac{1}{3}(\text{Ln}(z) + i2\pi)}$

$$\boxed{\therefore\ f_1(z) = e^{\frac{1}{3}(\text{Ln}(z) + i2\pi)}}$$

c) **(3 pts.)** Show that f_1 is analytic on D.

1. we know that $\text{Ln}(z)$ is analytic on $\{\mathbb{C} \setminus (-\infty, 0]\}$, and since $D = \{\text{Im}(z) > 0\}$ is a subset of $\{\mathbb{C} \setminus (-\infty, 0]\}$, we can say that $\text{Ln}(z)$ is also analytic on D.

2. since $\text{Ln}(z)$ is analytic on D, $e^{\frac{1}{3}\text{Ln}(z)}$ is also analytic on D because e^{w} is differentiable for any w & doesn't vanish on D.

3. Multiplying $e^{\frac{1}{3}\ln(z)}$ by a complex constant won't change it analyticity, so $e^{\frac{1}{3}\text{Ln}(z)} e^{\frac{1}{3}2\pi} = f_1(z)$ is analytic on D. ∎

3. (10 points) Show that the function

$$f(x,y) = \frac{x+1}{(x+1)^2+y^2} + i\left[\frac{-y}{(x+1)^2+y^2}\right]$$

satisfies the Cauchy-Riemann equations on $\mathbb{C}\backslash\{-1\}$.

1. $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x+1}{(x+1)^2+y^2}\right) = \frac{(x+1)^2+y^2-(x+1)\cdot 2(x+1)}{((x+1)^2+y^2)^2}$

$= \frac{y^2-(x+1)^2}{((x+1)^2+y^2)^2} \quad z \neq -1$

2. $\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left(\frac{-y}{(x+1)^2+y^2}\right) = \frac{-(x+1)^2-y^2-(-y)(2y)}{((x+1)^2+y^2)^2}$

$= \frac{y^2-(x+1)^2}{((x+1)^2+y^2)^2} \quad z \neq -1$

3. $\frac{\partial u}{\partial x} = \frac{y^2-(x+1)^2}{((x+1)^2+y^2)^2} = \frac{\partial v}{\partial y}$

4. $\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x+1}{(x+1)^2+y^2}\right) = -\frac{x+1}{((x+1)^2+y^2)^2}(2y) = \frac{-2y(x+1)}{((x+1)^2+y^2)^2}$

5. $-\frac{\partial v}{\partial x} = -\frac{\partial}{\partial x}\left(\frac{-y}{(x+1)^2+y^2}\right) = -\left(\frac{y}{((x+1)^2+y^2)^2}\right)(2(x+1)) = \frac{-2y(x+1)}{((x+1)^2+y^2)^2}$

6. $\frac{\partial u}{\partial y} = \frac{-2y(x+1)}{((x+1)^2+y^2)^2} = -\frac{\partial v}{\partial x}$

Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, Cauchy-Riemann equations is satisfied on $\mathbb{C}\backslash\{-1\}$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

4. (10 points) Let $u(x, y)$ be the function:

$$u(x, y) = \tan^{-1}(y/x)$$

on the domain $D := \{x + iy : x > 0\}$. Here, $\tan^{-1}$ denotes the real-valued arctangent function with range $(-\pi/2, \pi/2)$.

a) (**6 pts.**) Prove that u is harmonic on D.

✓

1. $u_x = \frac{1}{1+(\frac{y}{x})^2} \cdot \frac{-y}{x^2} = \frac{-y}{x^2+y^2}$. $u_{xx} = \frac{+y}{(x^2+y^2)^2}(2x) = \frac{2xy}{(x^2+y^2)^2}$, $x > 0$.

2. $u_y = \frac{1}{1+(\frac{y}{x})^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$ $u_{yy} = \frac{-x}{(x^2+y^2)^2}(2y) = \frac{-2xy}{(x^2+y^2)^2}$, $x > 0$

$\therefore \Delta u = u_{xx} + u_{yy} = \frac{2xy}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2} = 0 \Rightarrow$ harmonic

b) (**4 pts.**) Find a harmonic conjugate for u on D (that is, a function $v(x, y)$ on D such that $u + iv$ is analytic on D).

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1. $v_y = u_x = \frac{-y}{x^2+y^2} \Rightarrow \boxed{v = -\frac{1}{2}\ln(x^2+y^2)} + g(x)$.

$v_x = \frac{-x}{x^2+y^2} + g'(x) = -u_y = -\frac{x}{x^2+y^2}$

$\Rightarrow g'(x) = 0 \Rightarrow g(x) = C$

2. $v = -\frac{x}{x^2+y^2} + C$, let $C = 0$

$$\boxed{\therefore v(x,y) = \cancel{\frac{-x}{x^2+y^2}}}$$

5. (10 points) Show that the linear fractional transformation

$$f(z) = \frac{z-1}{z+1}$$

maps the unit circle $\{|z| = 1\}$ to the generalized circle $\{\mathrm{Re}(z) = 0\} \cup \{\infty\}$.

1. Under $f(z)$, we have

$$f(i) = \frac{i-1}{i+1} = \frac{(i-1)(i-1)}{(i+1)(i-1)} = \frac{\cancel{1}-2i+\cancel{1}}{-1-1} = i. \quad i \mapsto i$$

$$f(1) = \frac{1-1}{1+1} = 0. \quad 1 \mapsto 0$$

$$f(-i) = \frac{-i-1}{-i+1} = \frac{i+1}{i-1} = \frac{(i+1)(i+1)}{(i-1)(i+1)} = \frac{-1+2i+1}{-1-1} = -i. \quad -i \mapsto -$$

2. Since i, 1, and $-i$ lie on the unit circle, and that three points are enough to uniquely determine the image under LFT, we can say that the generalized circle $\{\mathrm{Re}(z) = 0\} \cup \{\infty\}$, which contains $f(i) = i$, $f(1) = 0$, $f(-i) = -i$, is mapped by the unit circle under the given LFT. ■

Illustrations: