1. Suppose f is a holomorphic function defined on a region U which is simply connected(that is, which satisfies the Poincare Lemma/ “p,q Theorem”). Show carefully using the Poincare Lemma that there is a holomorphic function F with $F' = f$ everywhere on U. (Do this directly. Do not appeal to the Cauchy Integral Theorem)

We look for F in the form $U + iV$, so that F is holomorphic if and only if $\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$ & $\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$.

Assuming F does satisfy those Cauchy - Riemann equations, $F' = f$ if $\frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = u + iv$ (since $F' = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x}$).

So F will have all the properties we need if

$$\frac{\partial U}{\partial x} = u = \frac{\partial V}{\partial y} \quad \text{and} \quad \frac{\partial V}{\partial x} = v = -\frac{\partial U}{\partial y}.$$

Let us look then for U and V separately. Can we arrange that $\frac{\partial U}{\partial x} = u$ and $\frac{\partial U}{\partial y} = -v$? Yes we can, by the "p,q theorem" since $\frac{\partial}{\partial y}(\overset{p}{u}) = \frac{\partial}{\partial x}(\overset{q}{-v})$ by the Cauchy - Riemann equations for $f = u + iv$.

Similarly we can find V such that

$\frac{\partial V}{\partial x} = \overset{p}{v}$ and $\frac{\partial V}{\partial y} = \overset{q}{u}$ because $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$ by the other CR equation for $f = u + iv$.

With U & V so chosen, $U + iV$ satisfies the CR equations and so is holomorphic and with $F = U + iV$, $F' = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = u + iv = f$. ✓

2(a) State the Cauchy Integral Formula
(b) Show how the Cauchy Integral Formula implies that a holomorphic function defined on the disc {z: |z| <1} has a power series expansion around 0 that converges to f on the same disc.

(a) f holomorphic on open set containing $\{z : |z-z_0| \le R\}$, $R>0$ then for each z with $|z-z_0| < R$,

$$f(z) = \frac{1}{2\pi i} \oint_{|\zeta - z_0| = R} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

Let $|z-z_0| = R_1 < R$

(b) Since $f(z) = \frac{1}{2\pi i} \oint_{|\zeta - z_0| = (R+R_1)/2} \frac{f(\zeta)}{(\zeta - z_0) - (z - z_0)}\, d\zeta$ and $\left|\frac{z-z_0}{\zeta - z_0}\right| < 1$

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(\zeta)}{\zeta - z_0} \cdot \frac{1}{1 - \left(\frac{z-z_0}{\zeta - z_0}\right)}\, d\zeta = \frac{1}{2\pi i} \oint \frac{f(\zeta)}{\zeta - z_0} \left(1 + \left(\frac{z-z_0}{\zeta - z_0}\right) + \left(\frac{z-z_0}{\zeta - z_0}\right)^2 + \cdots\right) d\zeta$$

integrating term by term

$$= \sum_{n=0}^{+\infty} (z-z_0)^n \left(\frac{1}{2\pi i} \oint_{|\zeta - z_0| = (R+R_1)/2} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta\right)$$

$$= \sum_{n=0}^{+\infty} (z-z_0) \frac{1}{n!} f^{(n)}(z_0) \quad \text{where } f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta$$

comes from diff under integral sign.

Since coefficients are independent of R_1, this expansion works for all z with $|z-z_0| < R$.

3 Find the negative power part and the first two nonzero nonnegative power terms of the Laurent series around 0 of the function $1/(\sin z)$ which is valid on the punctured disc $\{z: 0<|z|<\pi\}$.

$$\frac{1}{\sin z} = \frac{1}{z - z^3/3! + z^5/5! - z^7/7!}$$

$$= \frac{1}{z}\,\frac{1}{1 - \frac{z^2}{3!} + \frac{z^4}{5!}\cdots} = \frac{1}{z}\,\frac{1}{1 - R(z)}$$

where $R(z) = z^2/3! - \frac{z^4}{5!} + \frac{z^6}{7!}\cdots$

So

$$\frac{1}{\sin z} = \frac{1}{z}\left(1 + R(z) + R^2(z) + \cdots\right)$$

$$= \frac{1}{z}\Big(1 + \frac{z^2}{3!} - \frac{z^4}{5!} + \text{higher than 4th powers}$$

$$+ \left(\frac{z^2}{3!}\right)^2 + \underbrace{\text{higher than 4th powers}}_{(\text{rest of } R^2(z))}$$

$$+ \text{higher than 4th powers from } R^3(z),\ R^4(z) \text{ etc}\Big)$$

$$= \frac{1}{z} + \frac{z}{3!} + z^4\left(\left(\frac{1}{3!}\right)^2 - \frac{1}{5!}\right) + \text{higher than 3rd } z \text{ powers}$$

$$= \frac{1}{z} + \frac{z}{6} + z^4\left(\frac{1}{36} - \frac{1}{120}\right) + \text{higher than 3rd } z \text{ powers}$$

$$= \frac{1}{z} + \frac{z}{6} + \frac{7}{360}z^4 + \text{higher than 3rd } z \text{ powers}$$

4 (a) Write down the integral formula for the coefficients of the Laurent series of a function f which is holomorphic on {z: 0<|z|<1}.

(b)Show that the negative power coefficients are all 0 if there is a constant M such that |f(z)| < M for all z with 0<|z|< 1.

(a) $$a_m = \frac{1}{2\pi i} \oint_{|z|=\varepsilon} f(\zeta)\, \zeta^{-m-1}\, d\zeta \quad , \text{any } 0<\varepsilon<1 .$$

(b) Under $|f(z)|<M$ hypothesis, $|f(\zeta)\zeta^{-m-1}| \le M\varepsilon^{-m-1}$ if $|\zeta|=\varepsilon$

$$|a_m| \le \frac{1}{2\pi} \cdot \underbrace{2\pi\varepsilon}_{\text{length}} \cdot \underbrace{M\varepsilon^{-m-1}}_{\substack{\max\limits_{|\zeta|=\varepsilon} |f(\zeta)\zeta^{-m-1}| \text{ is } \le M\varepsilon^{-m-1} \text{ as noted}}} = M\varepsilon^{-m}.$$

So if $m<0$, $-m>0$ and $|a_m| \le \lim\limits_{\varepsilon\to 0} M\varepsilon^{-m} = 0$.

All negative power coefficients $= 0$.

5 (a) Find the negative power part of the Laurent series of $1/(z-1)(z-2)^2$ around z=1
(b) Do the same around z=2
(c) Verify that the sum of the two items for (a) and (b) is $= 1/(z-1)(z-2)^2$
(d) Explain why the equality in part (c) is guaranteed by general principles(limits at infinity and so on).

(a) $\frac{1}{(z-1)(z-2)^2}$ has simple pole at $z=1$

$$\frac{1}{(z-1)(z-2)^2} = \frac{a_{-1}}{z-1} + \text{nonnegative powers of } (z-1)$$

So
$$a_1 = \lim_{z\to 1} (z-1)\frac{1}{(z-1)(z-2)^2} = \lim_{z\to 1}\frac{1}{(z-2)^2} = 1$$

(b) $$\frac{1}{(z-1)(z-2)^2} = \frac{a_{-2}}{(z-2)^2} + \frac{a_{-1}}{(z-2)} + \cdots$$ where $z=2$ is double pole

$$a_{-2} = \lim_{z\to 2} (z-2)^2 \frac{1}{(z-1)(z-2)^2} = \lim_{z\to 2}\frac{1}{z-1} = 1.$$

$$\frac{1}{(z-1)(z-2)^2} - \frac{1}{(z-2)^2} = \frac{1-(z-1)}{(z-1)(z-2)^2} = \frac{2-z}{(z-1)(z-2)} = +\frac{-1}{z-2}\cdot\frac{1}{z-1}$$

$$= \frac{1}{z-2}\cdot\frac{-1}{(z-2)+1} = \frac{-1}{(z-2)}(1 + \text{positive powers of } z-2)$$

So $\frac{a_{-1}}{z-2}$ term $= \frac{-1}{z-2}$ and $a_1 = -1$.

(c) $$\frac{1}{z-1} + \frac{1}{(z-2)^2} + \frac{-1}{(z-2)} = \frac{(z-2)^2 + (z-1) - (z-2)(z-1)}{(z-1)(z-2)^2}$$

$$= \frac{z^2-4z+4+z-1-z^2+3z-2}{(z-1)(z-2)^2} = \frac{1}{(z-1)(z-2)^2}$$

(d) ((c)) This had to work because

$$\frac{1}{(z-1)(z-2)^2} - \frac{1}{z-1} - \left(\frac{1}{(z-2)^2} - \frac{1}{z-2}\right) \quad (*)$$

is holomorphic everywhere (singularities at $z=1$ and $z=2$ are subtracted off by "principal part" terms) and clearly $\lim_{|z|\to+\infty}$ (whole expression) $=0$ since each individual term $\to 0$ as $|z|\to+\infty$.

So whole expression $(*)$ is constant by Liouville's Theorem ($\lim = 0$ at infinity clearly implies bounded!). And the constant must be 0 since $\lim = 0$ as $|z| \to +\infty$.

Alternative way to do part(c): around $z=2$:

$$\frac{1}{z-1} = \frac{1}{1+(z-2)} = 1-(z-2)+(z-2)^2 \cdots$$

So
$$\frac{1}{(z-1)(z-2)^2} = \underbrace{\frac{1}{(z-2)^2} - \frac{1}{z-2}}_{\text{negative power part.}} + 1 \cdots$$