1. Suppose f is a holomorphic function defined on a region U which is simply connected(that is, which satisfies the Poincare Lemma1 "p,q Theorem"). Show carefully using the Poincare Lemma that there is a holomorphic function F with F' = f everywhere on U. (Do this directly. Do not appeal to the Cauchy Integral Theorem)

Let $F = U + iV$. Then F is holomorphic if $\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$ and $\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$. ~~Also~~ ((Cauchy Riemann Equations). Also, if these holds, then $F' = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x}$. So $F' = f$ if $\frac{\partial U}{\partial x} = u$ and $\frac{\partial V}{\partial x} = v$. Thus CR equations for $U + iV$ then require $\frac{\partial U}{\partial y} = -v$ and $\frac{\partial V}{\partial y} = u$

Everything will be satisfied if we can find U, V with $\frac{\partial U}{\partial x} = u$, $\frac{\partial U}{\partial y} = -v$ and $\frac{\partial V}{\partial x} = v$, $\frac{\partial V}{\partial y} = u$.

~~p, q Theorem say~~ there is a (real) g with $\frac{\partial g}{\partial x} = p$, $\frac{\partial g}{\partial y} = q$ if $\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}$ (on a suitable region). Thus U exists with $\frac{\partial U}{\partial x} = u$ $\frac{\partial U}{\partial y} = -v$ because $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ ($p = u$, $q = -v$) by CR for f. Similarly V exists with $\frac{\partial V}{\partial x} = v$, $\frac{\partial V}{\partial y} = u$ ($p = u$, $q = v$) because $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$ (CR for f, the other equation). So U, V exist as required.

2(a) State the Cauchy Integral Formula
(b) Show how the Cauchy Integral Formula implies that a holomorphic function defined on the disc $\{z : |z| < 1\}$ has a power series expansion around 0 that converges to f on the same disc.

(a) f holomorphic on open set U containing a closed disc $\{z : |z - z_0| = R\}$ then

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta \quad \text{where } \gamma \text{ is}$$

the circle $\{\zeta : |\zeta - z_0| = R\}$ and $|z - z_0| < R$

(b) Choose arbitrary $R > 0$ with $R < 1$ $(z_0 = 0)$

Then, with $|z| < R$:

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{\zeta} \frac{1}{1 - \frac{z}{\zeta}}\, d\zeta$$

$$= \frac{1}{2\pi i} \oint \frac{f(\zeta)}{\zeta} \left(1 + \frac{z}{\zeta} + \frac{z^2}{\zeta^2} + \dots\right) d\zeta$$ since $\left|\frac{z}{\zeta}\right| < 1$ because $|z| < R$, $|\zeta| = R$.

Integrating term by term gives

$$f(z) = \sum_{n=0}^{+\infty} z^n \left(\frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta \right) = \sum_{n=0}^{+\infty} z^n \cdot \frac{1}{n!} f^{(n)}(0)$$

$\left(\frac{1}{2\pi i} \oint \frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta = \frac{1}{n!} f^{(n)}(0) \right.$ by differentiating under the integral sign in the Cauchy integral formula). Since the coefficients of the series are independent of $R (<1)$ one gets that $f(z) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} z^n$ for all z with $|z| < 1$.

3 The basic estimate on the absolute value of a line integral is that

$$\left|\oint_\gamma f(z)\,dz\right| \le \text{length}(\gamma)\cdot \max_{\text{on } \gamma}|f| \qquad \gamma:[a,b]\to\mathbb{C}$$

Explain why this estimate is valid(Suggestion: Recall the "trick" of multiplying by a complex number of absolute value 1 to make the integral real non-negative)

If $\oint_\gamma f(z)\,dz = 0$, the inequality is clear.

So assume $\oint_\gamma f \neq 0$ and choose θ real so that

$e^{-i\theta}\oint_\gamma f$ is positive real (always possible).

Then since $|e^{-i\theta}|=1$

$$\left|\oint_\gamma f\right| = \left|e^{-i\theta}\oint_\gamma f\right| = e^{-i\theta}\oint_\gamma f = \operatorname{Re}\oint_\gamma e^{-i\theta} f =$$

$$\operatorname{Re}\left(\int_a^b e^{-i\theta} f(\gamma(t))\cdot\gamma'(t)\,dt\right) = \int_a^b \operatorname{Re}\left(e^{-i\theta} f(\gamma(t))\cdot\gamma'(t)\right)dt$$

$$\le \int_a^b \left|e^{-i\theta} f(\gamma(t))\cdot\gamma'(t)\right| = \int_a^b \left|e^{-i\theta} f(\gamma(t))\right|\cdot|\gamma'(t)|\,dt$$

$$= \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,dt \le \max|f(\gamma(t))|\cdot\int_a^b|\gamma'(t)|\,dt$$

$$= \max_{\text{on }\gamma}|f|\cdot\text{length}(\gamma(t))$$

since $\gamma'(t) = x'(t) + i y'(t)$ so $|\gamma'(t)| = \sqrt{(x')^2+(y')^2}$

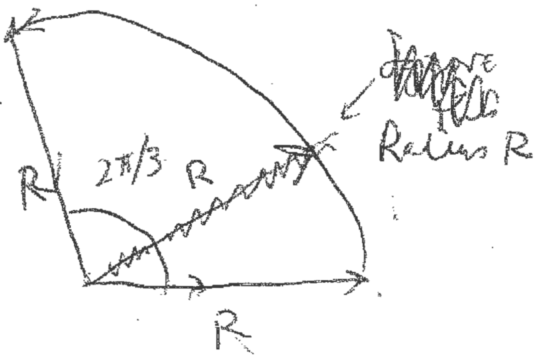
and $\int_a^b|\gamma'(t)|\,dt = \text{length of } \gamma.$

4 Find $$\int_0^{+\infty} \frac{1}{1+x^3}\,dx$$

by complex variables methods. Explain briefly but clearly why the steps in your calculation are valid.

$\int_0^{+\infty} \frac{1}{1+x^3}dx = \oint \frac{1}{1+z^3}dz$ limit as $R \to +\infty$. Look at

$\xrightarrow[0 \qquad\qquad R]{}$



$$\oint \frac{1}{1+z^3}dz = \oint_{\text{small circle around } e^{i\pi/3}} \frac{1}{1+z^3}dz = 2\pi i/(3z^2)\Big|_{e^{i\pi/3}} = \frac{2\pi i\, e^{i\pi/3}}{-1\cdot 3} = -\frac{2\pi i}{3}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$$

($e^{i\pi/3}$ is zero pt of denominator since $(e^{i\pi/3})^3 = -1$. no other zero pt is inside the contour)

$$\oint \frac{1}{1+z^3}dz = \oint \quad + \left(1-e^{2\pi i/3}\right)\int_0^R \frac{1}{1+x^3}dx$$

$\downarrow$

0 as $R \to +\infty$

Since $\left|\oint\right| \le \underbrace{\frac{1}{R^3-1}}_{\text{max (func)}} \cdot \underbrace{\frac{2\pi R}{3}}_{\text{length}} \to 0$ by basic estimate of integral

So $$\lim_{R\to+\infty}\int_0^R \frac{1}{1+x^3}dx = \frac{-\frac{2\pi i}{3}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)}{1-\left(\frac{\sqrt{3}}{2}i-\frac{1}{2}\right)} = \frac{\frac{2\pi i}{3}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)}{\frac{\sqrt{3}}{2}i-\frac{3}{2}}$$

$$= \frac{\frac{\pi}{3}\left(2\cdot\frac{1}{2}i + 2\frac{\sqrt{3}}{2}i\right)}{+\frac{3}{2}i+\frac{\sqrt{3}}{2}} = \frac{(\pi/3)(i+\sqrt{3}i)}{\frac{\sqrt{3}}{2}\cdot(\sqrt{3}i+i)} = \frac{\pi/3}{\sqrt{3}/2} = \frac{\pi/3}{\sin\left(\frac{\pi}{3}\right)} = \frac{2\pi}{3\sqrt{3}}$$

5 (a) Use the differentiated Cauchy integral formula and the result of problem 3 to show that a function f that is holomorphic on all of the complex plane C and which is bounded is necessarily constant.[f is bounded means there is an M>0 such that $|f(z)|\leq M$ for all complex numbers z]
(b) Explain briefly but clearly why part (a) implies that every nonconstant polynomial P(z) has a root, i.e., there is a z with P(z)=0.

(a) $$f'(z_0) = \frac{1}{2\pi i}\oint_{|\zeta - z_0| = R} \frac{f(\zeta)}{(\zeta - z_0)^2}\, d\zeta$$ by differentiating CI Formula

z_0, R arbitrary

So $$|f'(z_0)| \leq \frac{1}{2\pi}\,\underbrace{2\pi R}_{\text{length of curve}} \cdot \frac{\overset{\text{max } f}{M}}{\underset{|\zeta - z_0|^2}{R^2}} = \frac{M}{R}.$$

With z_0 fixed, let $R \to +\infty$ to show that $|f'(z_0)| = 0$.
So f' is 0 everywhere on $\mathbb{C}$ and f is thus constant
(since change in f from 0 to z $= \oint_{\text{line from 0 to } z} f' = \oint 0 = 0$)

(b) P nowhere zero $\Rightarrow$ $\frac{1}{P}$ holomorphic everywhere.
Since $|P(z)| \to +\infty$ as $|z| \to +\infty$ (highest degree term dominates). $1/P \to 0$ as $|z| \to \infty$. So $\left|\frac{1}{P}\right| \leq 1$ if $|z| > R_0$ for some $R_0 > 0$. Since $\left|\frac{1}{P}\right|$ is bounded for $|z| \leq R_0$, $\frac{1}{P}$ is bounded altogether. So $\frac{1}{P}$ is constant.
But $\lim_{|z| \to +\infty} \frac{1}{P(z)} = 0$ implies this constant is 0. but $\frac{1}{P(z)}$ is never 0, a contradiction. So $P(z) = 0$ somewhere.