Sample Problems for Final Math 132 Summer 2013

1. Explain how ~~the~~ the Laurent series of a function $f(z)$ holomorphic on $\{z : R_1 < |z| < R_2\}$ arises and find the formula for the coefficient of z^n ($n = 0, \pm 1, \pm 2, \ldots$) in terms of line integrals.

2. Use the formula of problem 1 to find the coefficients of $\frac{1}{z}$, $\frac{1}{z^2}$, and $\frac{1}{z^3}$ in the Laurent series for $1/\sin z$ that is valid for z satisfying $2\pi < |z| < 3\pi$.

3. Show the reason that $\left|\oint_\Gamma f(\zeta)\,d\zeta\right| < \text{Length}(\Gamma) \cdot \max_{z \text{ on } \Gamma} |f(z)|$ (involves multiplying by α, $|\alpha| = 1$ to make $\oint_\Gamma \alpha f$ positive real).

4. Use the estimate of problem 3 to show that the negative-power coefficients of the Laurent series of f holomorphic on $\{z : 0 < |z| < R\}$ $R > 0$ are 0 if f is bounded on $\{z : 0 < |z| < R\}$ (i.e. there is an $M > 0$ such that $|f(z)| \leq M$ for all z with $0 < |z| < R\}$.

5. Find $\int_0^{+\infty} \frac{1}{1+x^5}\,dx$ by the "pie slice" method.

6. Show that if f is holomorphic on a region (open set) containing $\{z : |z - z_0| \leq R\}$, $R > 0$, then the $\operatorname{Re} f(z_0)$ = the average of $\operatorname{Re} f$ on the circle $\{z : |z - z_0| = R\}$.

7. Show using the "p,q theorem" (Poincaré Lemma) that on a simply connected region every harmonic function u is the real part of some holomorphic function ("harmonic" means $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ everywhere).

8. Illustrate your construction in problem 7 by finding v such that $u + iv$ is holomorphic for $u = \frac{1}{2} \ln(x^2+y^2)$, region $= \{z = x+iy : x>0\}$.

9. Suppose $f = e^h$ for some holomorphic functions f and h. Show that $h' = f'/f$.

10. Use winding number (or Rouché's Theorem) arguments to show that all three zeroes of $3z^3 + z + 8$ satisfy $1 < |z| < 2$.

11. How many zeroes does $z^2 - z + 1$ have in each quadrant? Use winding numbers to show this!

12. Show that if f is not constant on a region U and $f(z_0) = w_0$, $z_0 \in U$, then there is a $\delta > 0$ such that, for every w with $|w - w_0| < \delta$, there is a z with $z \in U$ and $f(z) = w$. (Suggestion: Apply Rouché's Theorem with Γ a small circle around z_0 and $f - w_0$ and $f - w$ the two functions where $|w - w_0|$ on $\Gamma < \min_{\Gamma} |f - w_0|$).

13. Explain why if F is holomorphic on a region and Γ is a curve in the region then

$$\oint_\Gamma F' = F(\text{end of } \Gamma) - F(\text{beginning of } \Gamma)$$

[You may use the usual result from two-variable real calculus that $\oint_{\Gamma} \frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial y} dy$ $= h(\text{end}) - h(\text{beginning})$ for any function $h(x,y)$.]

14. Given f on a simply connected region (region where p,q Theorem / Poincaré Lemma works), f holomorphic, show that there is a holomorphic F with $F' = f$.

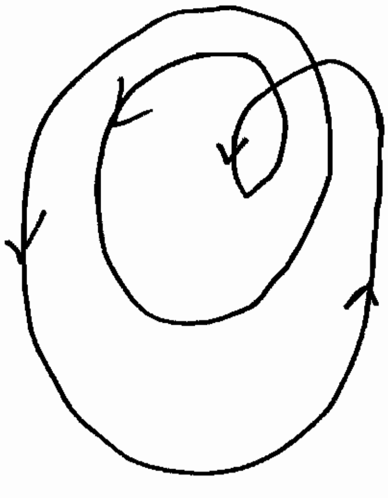
15. Combine 13 & 14 to deduce the Cauchy Integral Theorem.

16. Explain why if Γ is a simple closed curve in a region U on which f is holomorphic and if U contains the interior of Γ (as well as Γ), then

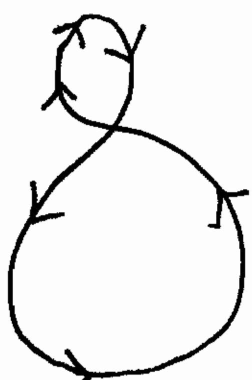
$$\frac{1}{2\pi i} \oint_\Gamma \frac{f'}{f} = \text{number of zeroes of } f \text{ inside } \Gamma \text{ counting orders}$$

17. Suppose f is holomorphic on an open set U with a finite number of points $p_1, \ldots, p_k$ removed. Explain why $f - \sum_k$ (neg power part of Laurent series around p_k) is holomorphic on all of U. Then discuss how this gives the "Residue Theorem" ($\oint_\Gamma f = 2\pi i$ (sum of residues at bad points inside)). This includes stating the Residue Theorem carefully!

19. Suppose f is holomorphic on $\{z : |z| < 1.1\}$ and that the image under f of the unit circle $|z|=1$ is the curve shown. Label by writing number the number of times $f(z)$, $|z|<1$ attains the values of various points.



20. Is it possible that the f-image curve of problem 19 could look like this? Why or why not?



21. Suppose f is holomorphic on $\{z : |z| < 1\}$ and $|f(z)| \le 1$ for all $|z|<1$ and $f(0)=0$ so $f(z)/z$ is holomorphic

(a) Show that $\left|\frac{f(z)}{z}\right| \le \frac{1}{R}$ if $|z| = R > 0$, $R<1$

(b) Deduce from Max. Modulus Principle that $\left|\frac{f(z)}{z}\right| \le \frac{1}{R}$ if $|z| \le R$, $R>0$, $R<1$.

(c) Conclude that $|f(z)/z| \le 1$ for all z with $|z|<1$

(d) Finally, conclude that $|f(z)| \le |z|$ all z with $|z|<1$ and that if $|f(z)| = |z|$ for some $z \neq 0$ then $f(z) = cz$.

22. State the Maximum Modulus Principle carefully and use the Open Mapping Theorem (prob. 12) to explain why it is true.

23. Suppose f is holomorphic on $\{z : |z| < 1.1\}$ and $|f(z)| = 1$ when $|z| = 1$. Show that

$$f(z) = c \left(\frac{z-a_1}{1-\bar{a}_1 z}\right)^{k_1} \cdots \left(\frac{z-a_n}{1-\bar{a}_n z}\right)^{k_n}$$

for some c with $|c| = 1$ and $a_1 \ldots a_n$ with $|a_j| < 1$ and positive integers $k_1 \ldots k_n$.

(Suggestion: Let $a_1 \cdots a_n$ be the zeroes of f with $|z| < 1$ and $k_1 \ldots k_n$ be their orders. Then

$f \Big/ \left(\frac{z-a_1}{1-\bar{a}_1 z}\right)^{k_1} \cdots \left(\frac{z-a_n}{1-\bar{a}_n z}\right)^{k_n}$ has $|\ \ | = 1$ when $|z| = 1$

and has no zeroes. This makes it constant! Explain why!].

24. Compute that $\left|\frac{z-a}{1-\bar{a}z}\right| < 1$ if $|z| < 1$, $|a| < 1$

and $= 1$ if $|z| = 1$ (still with $|a| < 1$).

25. The holomorphic function $\frac{z-a}{1-\bar{a}z}$ $|a| < 1$ has one simple zero inside the unit disc. Combine this with problem 24 and winding no arguments to deduce that the equation $\frac{z-a}{1-\bar{a}z} = w$ has exactly one solution for z, $|z| < 1$, for each given w with $|w| < 1$.

26. Find the "partial fraction decomposition" of $\frac{1}{(z-1)^2(z-3)^2}$ $\left(= \frac{A}{(z-1)} + \frac{B}{(z-1)^2} + \frac{C}{(z-3)} + \frac{D}{(z-3)^2}\right)$ expression as

and explain why it is bound to give

$\frac{1}{(z-1)^2(z-3)^2}$.

27. Find the partial fraction decomposition of $\frac{1}{1+z^4}$, and verify calculationally that it works.

28. Let P be a polynomial of degree n and first coefficient 1 : $z^n + a_{n-1}z^{n-1} + \dots + a_0$.

Show that the zeros of P are "stable under perturbation" in the following sense: Given $\varepsilon > 0$, there is a $\delta > 0$ such that if $Q(z) = z^n + b_{n-1}z^{n-1} + \dots + b_0$ with

$$|a_{n-1} - b_{n-1}| < \delta$$
$$\vdots$$
$$|a_0 - b_0| < \delta$$

then every zero of Q lies within ε of some zero of P. (Suggestion: Apply Rouché's Theorem to small circles around the zeroes of P).