University of California, Los Angeles
Department of Mathematics

Math 132: Complex Analysis for Applications, Winter 2013
Midterm 1
February 1, 2013, 2:00–2:50 pm

Last Name (please print):	**Solutions**
First Name (please print):	
University ID:	
Signature: **(do not sign before your ID is checked)**	
Instructor:	Y. van Gennip

Instructions:

1. **Do not open this booklet until told to do so.**
2. Fill in the above box. Please use the name under which you are registered.
3. This exam contains 5 pages with a total of 4 questions. Once the exam begins please check to make sure your exam is complete.
4. **Show all your work! Justify your answer unless it is specifically stated that you do not need to.**
5. If you run out of space in a problem, use the space on the back of the previous page and clearly indicate where the solution continues.
6. Calculators are **not** allowed.
7. No book, paper, or device, other than the usual writing instruments and this booklet shall be within reach of a student during the examination.
8. During the examination speaking to, communicating with, or deliberately exposing written papers to the view of other examinees is forbidden.
9. **Turn off your cellphones and other electronic devices.**
10. Try your best!

Do not write in this table!	
Question	**Points**
1	**/6**
2	**/6**
3	**/10**
4	**/8**
Total	**/30**

1. Compute the following. If your answer is a complex number, give it in either Cartesian or polar coordinates. If the answer is multi-valued give all possible values. **(Total: 6 points)**

   (a) $\arg\left(\dfrac{1+5i}{3+2i}\right) = \dots$

   Solution: First we can compute

   $$\frac{1+5i}{3+2i} = \frac{1+5i}{3+2i}\frac{3-2i}{3-2i} = \frac{13+13i}{13} = 1+i.$$

   Then we see that $\arg\left(\frac{1+5i}{3+2i}\right) = \arg(1+i) = \{\frac{\pi}{4} + 2\pi k : k \in \mathbb{Z}\}$.
   Some of you took another route:

   $$\begin{aligned}\arg\left(\frac{1+5i}{3+2i}\right) &= \arg(1+5i) - \arg(3+2i) \\ &= \{\arctan(5) - \arctan(2/3) + 2\pi k : k \in \mathbb{Z}\}.\end{aligned}$$

   This is correct, of course, but less informative, because it doesn't give the angles explicitly.

   (b) Find all complex numbers z which satisfy $z^5 = 1+i$.

   Solution: If we write in polar coordinates $z = re^{i\theta}$ and $i+1 = \sqrt{2}e^{i(\pi/4+2\pi k)}$, for $k \in \mathbb{Z}$, then the equation becomes

   $$r^5 e^{i5\theta} = \sqrt{2}e^{i(\pi/4+2\pi k)}.$$

   Hence $r = \sqrt[10]{2}$ and $\theta = \frac{1}{5}(\frac{\pi}{4} + 2\pi k)$ for $k \in \mathbb{Z}$. However, there are only five different values for k that give different answers. After that we get repetitions, because $e^{i2\pi} = 1$. So the five roots we get, are

   $$z_1 = \sqrt[10]{2}e^{i\pi/20}, \qquad z_2 = \sqrt[10]{2}e^{i9\pi/20}, \qquad z_3 = \sqrt[10]{2}e^{i17\pi/20},$$
   $$z_4 = \sqrt[10]{2}e^{i25\pi/20}, \qquad z_5 = \sqrt[10]{2}e^{i33\pi/20}.$$

   (c) $2\cos\big(5\log(3i)\big) - (3i)^{5i} = \dots$

   Solution: I realized too late that there are at least two different interpretations of the addition of multi-valued 'functions' you can apply here. The question is ambiguous, so I have given full points to any solution that showed insight in what was going on.
   From the definitions of the complex cosine and the complex power function we have

   $$2\cos(5\log 3i) = e^{i5\log 3i} + e^{-i5\log 3i},$$
   $$(3i)^{5i} = e^{5i\log 3i}.$$

(First interpretation.) We have

$$2\cos(5\log 3i) - (3i)^{5i} = e^{i5\log 3i} + e^{-i5\log 3i} - e^{5i\log 3i} = e^{-5i\log 3i}.$$

By the definition of the complex logarithm, we have, for $k \in \mathbb{Z}$

$$\begin{aligned} 2\cos(5\log 3i) - (3i)^{5i} &= e^{-i5\log 3i} = e^{-i5(\log 3 + i\pi/2 + i2\pi k)} \\ &= e^{5\pi/2 + 10\pi k} e^{-i5\log 3}. \end{aligned}$$

(Second interpretation.) We have, for $k, l, m \in \mathbb{Z}$,

$$\begin{aligned} 2\cos(5\log 3i) - (3i)^{5i} &= e^{i5(\log 3 + i\pi/2 + i2\pi k)} + e^{-i5(\log 3 + i\pi/2 + i2\pi l)} - e^{i5(\log 3 + i\pi/2 + i2\pi m)} \\ &= e^{-5\pi/2}(e^{-10\pi k} - e^{-10\pi m}) e^{5i\log 3} + e^{5\pi/2 + 10\pi l} e^{-5i\log 3}. \end{aligned}$$

Another reminder to be careful when working with multi-valued 'functions'. If you are interested in reading more about these kinds of problems with multi-valued 'functions', I came across an interesting article here:
http://www.researchgate.net/publication/
221352271_The_Challenges_of_Multivalued_Functions

2. Consider the three-dimensional unit sphere with the south pole at $(0, 0, -1)$ and the north pole at $(0, 0, 1)$. Let (X, Y, Z) denote the coordinates of points on the sphere. Determine the stereographic projection of the area on the sphere which satisfies $\frac{1}{2} \leq Z \leq \frac{3}{4}$ and sketch it. Justify your answer and compute relevant lengths in your sketch. **(6 points)**

   Solution: The region on the sphere for which $\frac{1}{2} \leq Z \leq \frac{3}{4}$ is bounded by two circles on the sphere. By the theorem in the book we know that circles on the sphere project to either circles or lines in the plane. Since the circles on the sphere do not go through the north pole, the projections are circles in the plane (so not straight lines). By symmetry the center of these circles is at the origin (0,0,0), so we only need to compute the radii.

   Let's first consider the projection of the circle on the sphere with $Z = \frac{3}{4}$. Let's call the radius of that circle a and call the radius of the projected circle, *i.e.*, the one we want to compute, r. If we look at the cross section of the sphere with the $x - z$ plane, we can draw two similar triangles: one with base of length r and height 1 (the radius of the sphere) and the other with base of length a and height $1 - \frac{3}{4} = \frac{1}{4}$. Because the triangles are similar, we know that $\frac{x}{a} = \frac{1}{1/4} = 4$. To compute a, we note that a is the length of a leg of a right triangle, whose other leg has length $\frac{3}{4}$ and whose hypothenuse has length 1. Thus $\sqrt{\left(\frac{3}{4}\right)^2 + a^2} = 1$. Because $a > 0$, we find $a = \sqrt{\frac{7}{16}}$ and thus $x = \sqrt{7}$.

   We can make a similar picture for thee projection of the circle with $Z = \frac{1}{2}$. In that case $\frac{x}{a} = \frac{1}{1/2} = 2$ and $\sqrt{\left(\frac{1}{2}\right)^2 + a^2} = 1$, so $a = \sqrt{\frac{3}{4}}$ and $x = \sqrt{3}$.

   We conclude that the projection is an annulus in the $x - y$ plane, centered at the origin, with inner radius $\sqrt{3}$ and outer radius $\sqrt{7}$.

   (I am going to be lazy, and will not produce the picture here.)

3. **(Total: 10 points)**

(a) For a function $f : \mathbb{C} \to \mathbb{C}$, give the definition of $f'(z)$ using the difference quotient.

Solution: $f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$.

(b) Use the definition from (a) to determine if the functions $g(z) = z^2$ and $h(z) = |z|^2$ are complex differentiable. If they are, compute the derivatives using the definition.

Solution: For g we compute

$$\frac{(z+\Delta z)^2 - z^2}{\Delta z} = \frac{2z\Delta z + (\Delta z)^2}{\Delta z} = 2z + \Delta z \to 2z, \qquad \text{as } \Delta z \to 0.$$

Thus g is complex differentiable and $g'(z) = 2z$.

For h we compute

$$\frac{|z+\Delta z|^2 - |z|^2}{\Delta z} = \frac{\overline{z}\Delta z + z\overline{\Delta z} + |\Delta z|^2}{\Delta z} = \overline{z} + z\frac{\overline{\Delta z}}{\Delta z} + \overline{\Delta z}.$$

If we approach the origin via the real axis, *i.e.*, we choose $\Delta z = \Delta x \in \mathbb{R}$, then

$$\overline{z} + z\frac{\overline{\Delta z}}{\Delta z} + \overline{\Delta z} = \overline{z} + z + \Delta x \to \overline{z} + z, \qquad \text{as } \Delta x \to 0.$$

However, if we approach via the imaginary axis, *i.e.*, we choose $\Delta z = i\Delta y$, for $\Delta y \in \mathbb{R}$, then

$$\overline{z} + z\frac{\overline{\Delta z}}{\Delta z} + \overline{\Delta z} = \overline{z} - z - i\Delta y \to \overline{z} - z, \qquad \text{as } \Delta y \to 0.$$

Both approaches lead to different values for the limit, so the limit for $\Delta z \to 0$ does not exist and h is not differentiable.

(c) Reproduce your results from part (b) using the Cauchy-Riemann equations.

Solution: If we write $f(x+iy) = u(x,y) + iv(x,y)$, then the Cauchy-Riemann equations are

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \end{cases}$$

If we check this for $g(z) = z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$, we get

$$\frac{\partial(x^2-y^2)}{\partial x} = 2x = \frac{\partial(2xy)}{\partial y} \quad \text{and} \quad \frac{\partial(x^2-y^2)}{\partial y} = -2y = -\frac{\partial(2xy)}{\partial x},$$

and thus g is complex differentiable and $g'(x+iy) = \frac{\partial(x^2-y^2)}{\partial x} + i\frac{\partial(2xy)}{\partial x} = 2x + i2y = 2z$.
For $h(z) = |z|^2 = x^2 + y^2$ we get

$$\frac{\partial(x^2+y^2)}{\partial x} = 2x \neq 0 = \frac{\partial(0)}{\partial y},$$

so h is not complex differentiable.

4. **(Total: 8 points)**

(a) Define a continuous branch of $z \mapsto \log z$ on the slit plane $\mathbb{C} \setminus \{z \in \mathbb{C} : \mathrm{Im}\,(z) = \mathrm{Re}\,(z) \geq 0\}$.

Solution: We have

$$\log z = \log |z| + i\theta.$$

We now need to define a range of values θ can take, such that the range starts and ends at the branch cut. So for example $\theta \in (-\frac{7\pi}{4}, \frac{\pi}{4})$. Of course other choices with multiples of 2π added to the endpoints of this interval work as well.

(b) Sketch the image of the vertical half line satisfying $\mathrm{Re}\,(z) = e$ and $\mathrm{Im}\,(z) \geq 0$ (minus the point on the line where $\mathrm{Re}\,(z) = \mathrm{Im}\,(z)$) under the branch you defined in (a). Back up your sketch with a computation.

Solution: I will use the branch I defined in (a), *i.e.*, I choose $\theta \in (-\frac{7\pi}{4}, \frac{\pi}{)}$. If you chose another range in (a), the computations have to be adapted.

For points z on the specified vertical half line, we can write $z = e + iy$. That leads to

$$\log z = \log \sqrt{e^2 + y^2} + i\theta,$$

where $\theta \in (-\frac{7\pi}{4}, -\frac{3\pi}{2}) \cup (0, \frac{\pi}{4})$. Note that θ ranges over two disjoint intervals, the first one corresponds to the part of the half line that lies above the branch cut, the other one to the part between the real axis and the branch cut.

For $\theta \in (0, \frac{\pi}{4})$ we can consider the right triangle with angle θ, base of length e, and hypotenuse of length $\sqrt{e^2 + y^2}$. This shows that

$$\sqrt{e^2 + y^2} = \frac{e}{\cos\theta}.$$

Hence, for the range $\theta \in (0, \frac{\pi}{4})$ we find

$$\log z = \log(e/\cos\theta) + i\theta = 1 - \log(\cos\theta) + i\theta.$$

For $\theta \in (-7\pi/4, -3\pi/2)$ we can do something similar, only now we have that $\theta = \varphi - 2\pi$, where now φ is the angle of a triangle with base of length e and hypotenuse of length $\sqrt{e^2 + y^2}$. As above we thus compute

$$\sqrt{e^2 + y^2} = \frac{e}{\cos\varphi} = \frac{e}{\cos(\theta + 2\pi)} = \frac{e}{\cos(\theta)},$$

and hence, also for $\theta \in (-7\pi/4, -3\pi/2)$ we conclude

$$\log z = \log(e/\cos\theta) + i\theta = 1 - \log(\cos\theta) + i\theta.$$

To help draw this, we can use that $1-\log(\cos(\frac{\pi}{4})) = 1-\log(\cos(-\frac{7\pi}{4}) = 1 - \log(\frac{1}{2}\sqrt{2}) = 1 + \log(\sqrt{2}) > 1 = 1 - \log(\cos 0)$ and $1 - \log(\cos\theta) \to \infty$ as $\theta \uparrow \frac{-3\pi}{2}$.

(Again I am being lazy, and I will not produce the picture here.)