UCLA

Department of Mathematics

Math 132 §2

Winter 2012

# Midterm #2

Attempt all problems. It is again noticed that the point total appears to be *greater* than 100; you will receive all points earned *unless* your score exceeds 100. Then you just get 100. Good luck to all, make your answers clear and concise (and a pleasure to grade). Justify your answers; right answers with little or no derivation will not necessarily receive full credit.

| | |
|---|---|
| Question #1 | 25 / 25 |
| Question #2 | 10 / 20 |
| Question #3 | 15 / 15 |
| Question #4 | 6 / 30 |
| Question #5 | 0 / 20 |

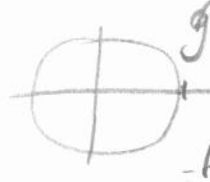
| | |
|---|---|
| Total | 56 / 100 |

Name: [redacted] ___ ___

(Please Print) – Last – – First – M.I.

Student ID Number: [redacted]

**Problem (1)** (25 Points) In the following, $f(z)$ is a function that is entire (analytic for all $z$) while $\alpha$ and $\beta$ are complex numbers with certain specifications that will be spelled out in each part of the question. We are going to integrate $f(z)[(z-\alpha)(z-\beta)]^{-1}$ around the unit circle ($|z|=1$) in the standard, counterclockwise fashion. In each case, you are going to evaluate the integral expressing your answer in terms of things like $\alpha$, $\beta$, $f(\alpha)$, $f''(\beta)$ etc. In at least one case, the said integral will be ill posed in which case you are to write "problem ill posed". In all cases, a correct answer will recieve full credit. However, you will not receive very much partial credit for a wrong answer with no justification. You may find it helpful to draw some pictures. Watch your factors of $2\pi i$.



$\oint = \frac{f(z)}{(z-\alpha)(z-\beta)}$

$= f(z)\left(\frac{A}{z-\alpha} + \frac{B}{z-\beta}\right)$

$\frac{A}{z-\alpha} + \frac{B}{z-\beta} = \frac{1}{(z-\beta)(z-\alpha)}$

$A(z-\beta) + B(z-\alpha) = 1$

$(A+B)z - A\beta - \alpha B = 1$

$A+B=0$

$-A\beta - \alpha B = 1$

$-A\beta + \alpha A = 1$

$A = \frac{1}{\alpha-\beta}$

$B = \frac{-1}{\alpha-\beta}$

[Part A, 5 points.] $|\alpha| > 1$, $|\beta| > 1$.

$\boxed{=0}$

[Part B, 5 points.] $|\alpha| < 1$, $|\beta| < 1$; $\beta \neq \alpha$.

$= \oint_\Gamma f(z)\left(\left(\frac{1}{\alpha-\beta}\right)\frac{1}{z-\alpha} + \left(\frac{-1}{\alpha-\beta}\right)\frac{1}{z-\beta}\right)dz$

$= \frac{f(\alpha)}{\alpha-\beta}\cdot 2\pi i - \frac{f(\beta)}{\alpha-\beta}2\pi i$

$= \boxed{\frac{2\pi i}{\alpha-\beta}\left(f(\alpha)-f(\beta)\right)}$

[Part C, 5 points.] $|\alpha| < 1$, $|\beta| = 1$.

problem ill-posed

[Part D, 5 points.] $|\alpha| < 1$, $\beta = \alpha$.

$= \oint_\Gamma \frac{f(z)}{(z-\alpha)^2}dz = \boxed{f'(\alpha)\cdot 2\pi i}$

[Part E, 5 points.] $|\alpha| < 1$, $|\beta| > 1$.

$= \oint_\Gamma f(z)\left(\frac{1}{\alpha-\beta}\frac{1}{z-\alpha} - \frac{1}{\alpha-\beta}\frac{1}{z-\beta}\right)dz$

$= \frac{f(\alpha)}{\alpha-\beta}2\pi i - 0 = \boxed{\frac{f(\alpha)2\pi i}{\alpha-\beta}}$

**Problem (2)** (20 Points). Let $A \in \mathbb{R}$ satisfy $|A| > 1$.

Part A (10 points). Show that

$$\operatorname{Re}\left[\frac{1}{A+e^{i\theta}}\right] = \frac{A+\cos\theta}{A^2+2A\cos\theta+1}$$

$$Re\left(\frac{1}{A+e^{i\theta}}\right) = Re\left(\frac{1}{A+\cos\theta+i\sin\theta}\right)$$

$$= Re\left(\frac{A+\cos\theta-i\sin\theta}{(A+\cos\theta+i\sin\theta)(A+\cos\theta-i\sin\theta)}\right) =$$

$$= Re\left(\frac{A+\cos\theta-i\sin\theta}{(A+\cos\theta)^2+\sin^2\theta}\right) = Re\left(\frac{A+\cos\theta-i\sin\theta}{A^2+2A\cos\theta+\cos^2\theta+\sin^2\theta}\right)$$

$$= Re\left(\frac{A+\cos\theta}{A^2+2A\cos\theta+1} + i\,\frac{-\sin\theta}{A^2+2A\cos\theta+1}\right) = \frac{A+\cos\theta}{A^2+2A\cos\theta+1}$$

Part B (10 points). Using your result from Part A of this problem, compute, quickly and easily

$$\int_0^{2\pi} \frac{A+\cos\theta}{A^2+2A\cos\theta+1}\, d\theta.$$

$$\int_0^{2\pi} \frac{A+\cos\theta}{A^2+2A\cos\theta+1}\, d\theta = Re\int_0^{2\pi}\frac{1}{A+e^{i\theta}}\,d\theta \quad . \text{ Let } z = e^{i\theta} \ \log$$

$$\frac{dz}{d\theta} = ie^{i\theta} = iz$$

$$d\theta = \frac{1}{iz}\,dz$$

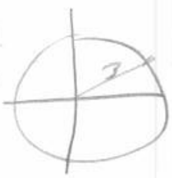
$$= Re\int_{z=e^{i0}}^{z=e^{i2\pi}} \frac{dz}{iz(A+z)} = Re\,\frac{1}{i}\int_0^0 \frac{dz}{Az+z^2} = \boxed{0}$$

**Problem (3)** (15 Points). Let $\Gamma$ be a circular contour of radius 3 centered at the origin. Compute

$$\mathrm{Re}\left[\oint_\Gamma \frac{e^{\lambda z}}{(z-2i)^3}dz\right]$$

where $\lambda$ is a real number.

$f = e^{\lambda z}$ is analytic everywhere because it is defined, smooth, & differentiable everywhere



$$Re\left[\oint_\Gamma \frac{e^{\lambda z}}{(z-2i)^3}dz\right] = Re\left(\frac{d^2}{dz^2}\left(e^{\lambda z}\right)\Big|_{z=2i} \cdot \frac{2\pi i}{2!}\right)$$

$$= Re\left[\lambda^2 e^{\lambda z}\Big|_{z=2i} \cdot \pi i\right]$$

$$= Re\left[\lambda^2 e^{2i\lambda} \cdot \pi i\right]$$

$$= Re\left(\lambda^2 \pi i\left(\cos(2\lambda) + i\sin(2\lambda)\right)\right)$$

$$= Re\left(\lambda^2 \pi i\cos 2\lambda - \lambda^2\pi\sin 2\lambda\right)$$

$\checkmark$ 15

$$= \boxed{-\lambda^2\pi\sin(2\lambda)}$$

**Problem (4)** (30 Points) Let $f(z)$ be a function – which we may as well call $z^{3/2}$ – and which has the following properties:

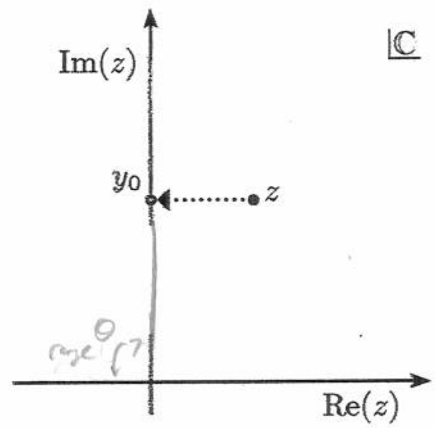
(i) For $z$ positive and real: $z = x_0$, $x_0 > 0$, $f(x_0)$ is the usual $x_0^{3/2}$.

(ii) The function $f(z)$ is analytic everywhere except the origin and the positive imaginary axis (where the function is not defined). Explicitly, we are using the convention

$$-\frac{3}{2}\pi < \theta < \frac{1}{2}\pi.$$

Part A [10 points]. Let $y_0 > 0$ be a generic positive number and $\varepsilon \ll 1$ also positive. Compute (see picture):

$$\lim_{\varepsilon \to 0} f(\varepsilon + iy_0).$$




$$\lim_{\varepsilon \to 0} f(\varepsilon + iy_0) = \lim_{\varepsilon \to 0} (\varepsilon + iy_0)^{3/2}$$

$$= \lim_{\varepsilon \to 0} \sqrt{\varepsilon^3 + 3\varepsilon^2 iy_0 - 3\varepsilon y_0^2 - iy_0^3}$$

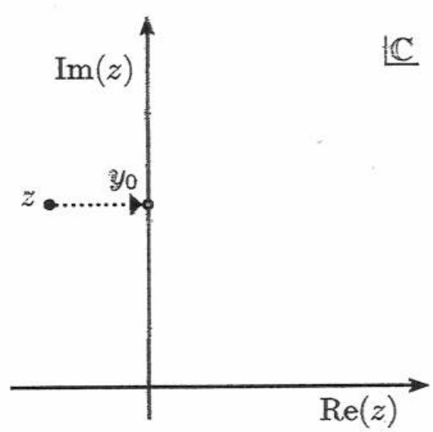
$$= \boxed{\sqrt{-iy_0^3}}$$

2

(In retrospect, I should have probably used the exponent formula for complex numbers.)

Part B [10 points]. Let $y_0 > 0$ be a generic positive number and $\varepsilon \ll 1$ also positive. Compute (see picture):

$$\lim_{\varepsilon \to 0} f(-\varepsilon + iy_0).$$




$$\lim_{\varepsilon \to 0} f(-\varepsilon + iy_0) = \lim_{\varepsilon \to 0} (-\varepsilon + iy_0)^{3/2}$$

$$= \lim_{\varepsilon \to 0} \sqrt{(-\varepsilon)^3 + 3(-\varepsilon)^2 iy_0 + 3(-\varepsilon) y_0^2 - iy_0^3}$$

$$= \lim_{\varepsilon \to 0} \sqrt{-\varepsilon^3 + 3\varepsilon^2 iy_0 - 3\varepsilon y_0^2 - iy_0^3}$$

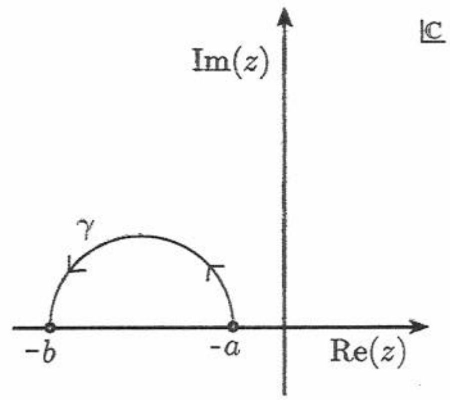
$$= \boxed{\sqrt{-iy_0^3}}$$

0

Certainly not same as above

Part C [10 points]. Let $a$ and $b$ denote positive numbers with $a < b$ and consider the points $z = -a$ and $z = -b$ on the negative real axis. Let $\gamma$ denote the semicircular contour in the upper half plane that connects these points as shown. Let $f(z)$ denote the (very same) $z^{3/2}$–type function which was the subject of Part A and Part B of this question. Compute the complex integral

$$\int_\gamma f(z)dz.$$




$$\int_\gamma f(z)dz = \int_\gamma z^{3/2}\,dz$$

$\gamma$ is a semicircle of radius $\frac{b-a}{2}$. A parameterization is $z = e^{i\theta}\left(\frac{b-a}{2}\right)$, $\theta \in [0,\pi]$

$\frac{dz}{d\theta} = i\,\frac{b-a}{2}\,e^{i\theta}$

$$= \int_0^\pi \left(\frac{b-a}{2}e^{i\theta}\right)^{3/2} i\,\frac{b-a}{2}\,e^{i\theta}\,d\theta$$

$$= \int_0^\pi i\left(\frac{b-a}{2}\right)^{3/2+1} e^{i\theta\cdot 3/2 + i\theta}\,d\theta$$

$$= i\left(\frac{b-a}{2}\right)^{5/2}\int_0^\pi e^{i\theta\, 5/2}\,d\theta$$

$e^{i\pi} = -1$

$(-1)^{5/2} = i^5 = i$

$$= i\left(\frac{b-a}{2}\right)^{5/2}\left[\frac{2}{5i}e^{i\theta 5/2}\right]_0^\pi = i\left(\frac{b-a}{2}\right)^{5/2}\left(\frac{2}{5i}\right)\left(e^{i\pi 5/2} - e^0\right)$$

$$= i\left(\frac{b-a}{2}\right)^{5/2}\left(\frac{2}{5i}\right)(i-1)$$

$$= \boxed{(1-i)\left(\frac{2}{5}\right)\left(\frac{b-a}{2}\right)^{5/2}}$$

4

Use antiderivative.

**Problem (5)** (20 Points) Let $H > 0$ be a positive number and consider the contour $\Gamma_H = \Gamma_{H_1} \cup \Gamma_{H_2}$ consist of two straight lines: $\Gamma_{H_1}$ connects the point $z = 1$ (on the $x$–axis) to the point $z = iH$ (on the $y$–axis) and $\Gamma_{H_2}$ connects $z = iH$ to the origin (going straight down the $y$ axis). Thus $\Gamma_H$ may be regarded as two legs of a right triangle.

Let $f(z) = ze^{-iz}$. Show regardless of $H$ (i.e., even if $H$ is "huge") that

$$\left|\int_{\Gamma_H} f(z)dz\right| \leq 1.$$


Note: this bound can be improved. And doing so *may* get you a bonus point or two.

$$\left|\int_{\Gamma_H} f(z)dz\right| = \left|\int_{\Gamma_H} ze^{-iz}dz\right| = \left|\int_{\Gamma_{H_1}} ze^{-iz}dz + \int_{\Gamma_{H_2}} ze^{-iz}dz\right| \leq \left|\int_{\Gamma_{H_1}} ze^{-iz}dz\right| + \left|\int_{\Gamma_{H_2}} ze^{-iz}dz\right|$$

by the Triangle Inequality

$$\left|\int_{\Gamma_H} ze^{-iz}dz\right| \leq$$

$$|\Gamma_H| = \sqrt{(H)^2 + 1^2} + H = H + \sqrt{1+H^2}$$

max of $ze^{iz}$:

$$|e^{-iz}| = \left|\frac{1}{e^{iz}}\right| = \frac{1}{|e^{iz}|} = \frac{1}{e^x} \leq 1$$

$$|z| = H$$

$$|e^{-iz}||z| \leq H$$

$$\left|\int_{\Gamma} ze^{-iz}dz\right| \leq \left(\sqrt{1+H^2} + H\right)(H)$$

$$\lim_{H\to\infty} \left(\sqrt{1+H^2} + H\right)(H) =$$

$$\leq \lim_{H\to\infty} (H + iH)(H)$$

?