

UCLA



Math 132§3 Spring 2010

Department of Mathematics

Midterm #2

Attempt all problems. As before: although the point total actually adds up to more than 100 you will receive all the points you earn up to 100 points after which you just get 100. Please note that certain problems are *seemingly* impossible but turn out to not too hard with Math 132 techniques. Good luck to one and all, make your answers clear and concise (and EZ to grade). Justify your answers; right answers with little or no derivation will not necessarily receive full credit.

Question #1	5 / 30
Question #2	5 / 20
Question #3	10 / 20
Question #4	5 / 20
Question #5	5 / 20
Total	52 / 100

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Problem (1) (30 Points) In the following, $f(z)$ is a function that is entire (analytic for all z) while α and β are complex numbers with certain specifications that will be spelled out in each part of the question. We are going to integrate $f(z)[(z-\alpha)(z-\beta)]^{-1}$ around the unit circle ($|z|=1$) in the standard, counterclockwise fashion. In each case, you are going to evaluate the integral expressing your answer in terms of things like $\alpha, \beta, f(\alpha), f''(\beta)$ etc. In at least one case, the said integral will be ill posed in which case you are to write "problem ill posed". In all cases, a correct answer will receive full credit. However, you will not receive very much partial credit for a wrong answer with no justification. You may find it helpful to draw some pictures. Watch your factors of $2\pi i$.

[Part A, 6 points.] $|\alpha| > 1, |\beta| > 1$.

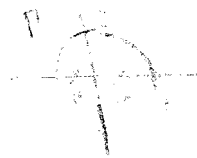
Pl. 1 $\oint \frac{f(z)}{(z-\alpha)(z-\beta)} dz = \left[\frac{f(z)}{z-\alpha} \right]_{z=\beta}$

$z = \alpha, z = \beta$ - direction of integration

[Part B, 6 points.] $|\alpha| < 1, |\beta| < 1$.

problem ill posed

Both α and β are inside the unit circle



inside the unit circle, so the integral cannot be solved

[Part C, 6 points.] $|\alpha| < 1, |\beta| = 1$.

Pl. 1 $\oint \frac{f(z)}{(z-\alpha)(z-\beta)} dz = \left[\frac{f(z)}{z-\alpha} \right]_{z=\beta}$

$= \oint \frac{f(z)}{z-\alpha} dz$

[Part D, 6 points.] $|\alpha| < 1, \beta = \alpha$.

α is inside the contour and so is β because $\beta = \alpha$, thus the integral cannot be solved

(Also, the answer could have a β in the denominator, making the problem ill posed)

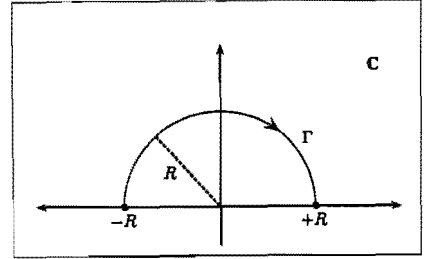
[Part E, 6 points.] $|\alpha| < 1, |\beta| > 1$.

Pl. 1 $\oint \frac{f(z)}{(z-\alpha)(z-\beta)} dz = \oint \frac{f(z)}{z-\alpha} dz = \left[\frac{f(z)}{z-\alpha} \right]_{z=\alpha}$

Problem (2) (20 Points) Consider the functions

$$P(x, y) = e^x \cos y, \quad Q(x, y) = e^x \sin y$$

and let Γ denote the semi-circular contour of radius R in the upper half plane starting at $z = -R$ and ending at $z = +R$ which is depicted to the right. Compute the integral



$$\int_{\Gamma} P dx - Q dy.$$

$$= \int_{\Gamma} f(z) dz$$

$$= \int_{\Gamma} e^z dz$$

e^z is entire, so any contour integral is path independent.

$$= \int_{-R}^{+R} e^x dx = \left[e^x \right]_{-R}^{+R} = e^R - e^{-R}$$

So $e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y)$
 for $z = x + iy$

Problem (3) (20 Points) In the following, we are to construct a function, $f(z)$ which we may call $z^{\frac{1}{4}}$: " $f(z) = z^{\frac{1}{4}}$ ". Of course there are many candidates for such a function; ours will satisfy the following criteria:

- For $\text{Im}(z) = 0, \text{Re}(z) = x > 0, f(z) = x^{\frac{1}{4}}$.
- For all z except $\text{Im}(z) > 0, f(z)$ is analytic.

Part A (5 points). Provide an unambiguous formula for such a function valid everywhere (except, possibly, the positive imaginary axis).

Handwritten notes for Part A:

$z = re^{i\theta}$
 $\sqrt[4]{z} = \sqrt[4]{r} e^{i\theta/4}$
 $0 \leq \theta < 2\pi$

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↑ this disc. on pos. real axis

Part B (5 points). For $z = 81$ (i.e. $81 + 0i$), compute $f(z)$.

Handwritten solution for Part B:

$$f(81) = 81^{1/4} e^{i0/4} = 81^{1/4} = 3$$

Handwritten note: "in disc. of real part of z, positive imaginary axis, $0 \leq \theta < 2\pi$ "

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Part C (5 points). For $z = -81$ (i.e. $-81 + 0i$), compute $f(z)$.

Handwritten solution for Part C:

$$f(-81) = (81)^{1/4} e^{i\pi/4} = 3(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}) = 3 \left(\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right)$$

(Note: The handwritten expression is crossed out with a large X.)

Part D (5 points). For $z = -i81$ (i.e. $0 - 81i$), compute $f(z)$.

Handwritten solution for Part D:

$$f(-i81) = 81^{1/4} e^{-i\pi/4} = 3 \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right) = 3 \left(-\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right)$$

(Note: The handwritten expression is crossed out with a large X.)

Problem (4) (20 Points) Consider the complex valued function J of the real variable θ given by

$$J(\theta) = \frac{1}{2 + i + e^{i\theta}}$$

Part A (10 points). Writing $J(\theta) = K(\theta) + iL(\theta)$, find expressions for K and L .

$$J(\theta) = \frac{1}{2 + i + \cos\theta + i\sin\theta} = \frac{1}{2 + \cos\theta + i(1 + \sin\theta)}$$

$$= \frac{2 + \cos\theta - i(1 + \sin\theta)}{(2 + \cos\theta + i(1 + \sin\theta))(2 + \cos\theta - i(1 + \sin\theta))} = \frac{2 + \cos\theta - i(1 + \sin\theta)}{4 + 4\cos\theta + \cos^2\theta + 1 + 2\sin\theta + \sin^2\theta}$$

$$= \frac{2 + \cos\theta - i(1 + \sin\theta)}{5 + 4\cos\theta + \sin^2\theta + \cos^2\theta} = \frac{2 + \cos\theta - i(1 + \sin\theta)}{6 + 4\cos\theta + 2\sin^2\theta}$$

$$K(\theta) = \frac{2 + \cos\theta}{6 + 4\cos\theta + 2\sin^2\theta}$$

$$L(\theta) = -\frac{1 + \sin\theta}{6 + 4\cos\theta + 2\sin^2\theta}$$

Part B (10 points). Compute $\int_0^{2\pi} K(\theta) d\theta$ where $K(\theta)$ is the function mentioned in Part A above.

$$\int_0^{2\pi} K(\theta) d\theta = \text{Re} \left(\int_0^{2\pi} \frac{1}{2 + i + e^{i\theta}} d\theta \right) = \text{Re} \left(\oint_{|z|=1} \frac{1}{z(2 + i + z)} dz \right)$$

Let $A(z) = \frac{1}{z(2 + i + z)}$

$A(z)$ is only defined when $z \neq 0, -2-i$. However, $-2-i$ is not within or on the contour. Thus,

$$\oint_{|z|=1} A(z) dz = 0$$

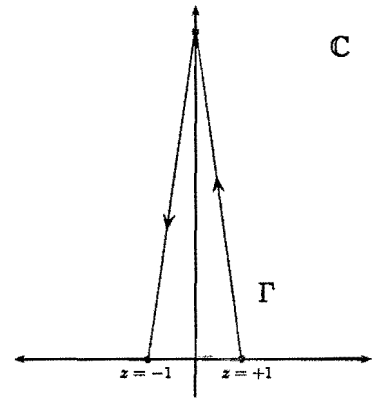
(2)

Re(0) = 0

Problem (5) (20 Points) Let Γ denote the contour which starts at $z = +1$ goes to $z = 10i$ - along a straight line - and then down to $z = -1$ - along a straight line. This is depicted to the right.

Consider the function e^{-z^2} and show

$$\left| \int_{\Gamma} e^{-z^2} dz \right| \leq 2.$$



$$\left| \int_{\Gamma} e^{-z^2} dz \right| \leq M \cdot \text{length}$$

$$\text{length}(\Gamma) = 2\sqrt{101}$$

M. maximum of $|e^{-z^2}|$ along Γ

$$|e^{-z^2}| = \frac{1}{|e^{z^2}|} = \frac{1}{|e^{x^2 - y^2 + 2ixy}|} = \frac{1}{e^{x^2 - y^2} \cdot |e^{2ixy}|}$$

$$\frac{1}{e^{x^2 - y^2} \cdot |e^{2ixy}|} = \frac{1}{e^{x^2 - y^2} \cdot 1} = \frac{1}{e^{x^2 - y^2}}$$

$\frac{1}{e^{x^2 - y^2}}$ ← this is largest when $x^2 - y^2$ is smallest
 smallest of course is 0. This occurs at $x = 1, y = 0$.

$$M = \frac{1}{e}$$

$$\left| \int_{\Gamma} e^{-z^2} dz \right| \leq \frac{2\sqrt{101}}{e}$$

and say we should have made \dots

M should be $\frac{1}{\sqrt{101}}$