UCLA

Department of Mathematics

Math 132 §2
Winter 2012

Midterm #1

Attempt all problems. It is noticed that the point total appears to be *greater* than 100. You will receive all points earned unless your score exceeds 100. Then you just get 100. Good luck to all, make your answers clear and concise (and a pleasure to grade). Justify your answers; right answers with little or no derivation will not necessarily receive full credit.

Question #1	~~5~~ / 10	10 (asked for a re-grade since there was an error grading it)
Question #2	15 / 20	
Question #3	11 / 20	
Question #4	0 / 20	
Question #5	11 / 20	
Question #6	20 / 20	

Total	~~67~~ / 100	67

Name: [redacted] ___
(Please Print) – Last – – First – M.I.

Student ID Number: [redacted]

Problem (1) (10 Points) The following five numbers are all expressed as complex exponentials. Write them in the form $a + ib$ with a and b real. The quantities a and b must be expressed as ratios of numbers, square roots, etc. You will get two points each for each right answer and loose three points for each wrong answer. Having a negative number of points is considered a mathematical impossibility so the worst that you can get on this question is a zero. But: leaving a question blank is considered a wrong answer.

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Part (a) (2 points).

$$e^{-i10\pi} = \cos(-10\pi) + i\sin(-10\pi)$$
$$= \cos(10\pi) - i\sin(10\pi)$$
$$= \boxed{1 + i0}$$

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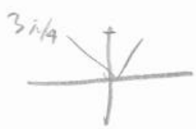
Part (b) (2 points).



$$e^{i\frac{3}{4}\pi} = \cos(3/4\,\pi) + i\sin(3/4\,\pi)$$
$$= \boxed{-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}}$$

==

Part (c) (2 points).

$$e^{-i\frac{\pi}{2}} = \cos(-\pi/2) + i\sin(-\pi/2)$$
$$= \cos\pi/2 - i\sin(\pi/2)$$
$$= \boxed{0 + i(-1)}$$

3 OK

==

Part (d) (2 points).

$$e^{i\pi} = \cos(\pi) + i\sin(\pi)$$
$$= \boxed{-1 + i0}$$

==

Part (e) (2 points).

$$e^{-\frac{3}{2}\pi i} = \cos(-3/2\,\pi) + i\sin(-3/2\,\pi)$$
$$= \cos(3\pi/2) - i\sin(3\pi/2)$$
$$= \boxed{(0) + i(1)}$$

==

Problem (2) (20 Points)

Part (a) 10 points. Let $z = 10 - 10i$ Write z in polar form ($z = re^{i\theta}$ with $-\pi < \theta \leq +\pi$).

$\theta = \tan^{-1}\left(\frac{-10}{10}\right) = \tan^{-1}(-1) = -\pi/4$

$r = \sqrt{10^2 + (-10)^2} = \sqrt{100+100} = \sqrt{200} = 10\sqrt{2}$

$\boxed{z = 10\sqrt{2}\, e^{-\pi/4\, i}}$

✓ 10

Part (b) (10 points). Let

$$z = \frac{e^{i\alpha}}{2 - e^{i\alpha}}$$

$\frac{z'}{2-z'}$ $\quad \overline{2-z'} = 2 - \bar{z}'$

with α a real number. Write z in the for $a + ib$ with a and b real.

5/10

$\frac{e^{i\alpha}}{2-e^{i\alpha}} = \frac{\cos\alpha + i\sin\alpha}{2 - \cos\alpha - i\sin\alpha} = \frac{(\cos\alpha + i\sin\alpha)(2 - \cos\alpha + i\sin\alpha)}{(2-\cos\alpha)^2 + (i\sin\alpha)^2}$

algebra errors.

$= \frac{2\cos\alpha + 2i\sin\alpha - \cos^2\alpha - \cos\alpha\, i\sin\alpha + i\sin\alpha\cos\alpha - \sin\alpha\cos\alpha}{4 - 4\cos\alpha + \cos^2\alpha - \sin^2\alpha}$

-5

$= \boxed{\frac{2\cos\alpha - \cos^2\alpha - \sin\alpha\cos\alpha}{4 - 4\cos\alpha + \cos^2\alpha - \sin^2\alpha} + i\,\frac{2\sin\alpha}{4 - 4\cos\alpha + \cos^2\alpha - \sin^2\alpha}}$

Problem (4) (20 Points) For the ordinary (real) trigonometric function $\cos x$, there is the famous identity

$$\cos^2 x = \frac{1}{2}[\cos 2x + 1].$$

Now consider the definition of $\cos z$ (with $z = x + iy$) as a function of the complex variable z.

Part (a) (5 points). Does the (analog of the) displayed identity hold for *all* complex values of argument?

YES NO (Circle one)

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Part (b) (15 points). Provide justification for your answer in Part (a). Specifically, if you said YES, provide a full derivation or if you said NO, find a value $z \in \mathbb{C}$ for which the generalization of the above identity fails.

$$\cos^2(z) = \cos(z)^2 = \cos(x+iy)^2 = \left[\frac{e^{(x+iy)i} + e^{-(x+iy)i}}{2}\right]^2$$

$$= \frac{1}{4}\left(e^{-y+xi} + e^{y-xi}\right)^2 = \frac{1}{4}\left(e^{(-y+xi)^2} + 2e^{(-y+xi)(y-xi)} + e^{(y-xi)^2}\right)$$

$$= \frac{1}{4}\left(e^{y^2-2xyi-x^2} + 2e^{-y^2+2yxi+x^2} + e^{y^2-2yxi-x^2}\right) = \frac{1}{2}\left(e^{y^2-2xyi-x^2} + e^{-y^2+2yxi+x^2}\right)$$

$$= \frac{1}{2}\left(e^{(y+xi)^2} + e^{(x+yi)^2}\right)$$

$$\frac{1}{2}\left[\cos(2z)+1\right] = \frac{1}{2}\left(\frac{e^{(x+iy)i} + e^{-(x+iy)i}}{2} + 1\right) = \frac{1}{4}\left(e^{xi-y} + e^{-xi+y} + \frac{1}{2}\right)$$

Let $x = 0$

Let $y = 3$

$$\frac{1}{2}\left(e^{y^2} + e^{-y^2}\right) = \frac{1}{2}\left(e^{9} + e^{-9}\right)$$

$$\neq$$

$$\frac{1}{4}\left(e^{-y} + e^{y} + \frac{1}{2}\right) = \frac{1}{4}\left(e^{-3} + e^{3} + \frac{1}{2}\right)$$

$\boxed{\text{If } z = 0+3i\text{, the identity fails}}$

Problem (5) (20 Points) Let $u(x,y)$ and $v(x,y)$ denote functions that are harmonic conjugates of one another. Explicitly, u and v satisfy $u_x = v_y$; $u_y = -v_x$. Now consider the function

$$U(x,y) = e^{[u^2(x,y) - v^2(x,y)]}\cos(2u(x,y)v(x,y)).$$

Q: Is $U(x,y)$ a harmonic function? Answer YES or NO then prove your assertion – i.e., a *full derivation* is required. Hint: This problem, viewed from a certain perspective, is quite easy.

$$U = e^{u^2-v^2}\cos(2uv) = \mathrm{Re}\{e^{u^2-v^2}e^{i2uv}\} = \mathrm{Re}\{e^{(u+vi)^2}\}$$

$$\frac{\partial U}{\partial x} = \mathrm{Re}\{e^{(u+vi)^2}2(u+iv)(u_x+v_x i)\} = \mathrm{Re}\{2e^{(u+vi)^2}(uu_x + uv_x i + u_x vi - vv_x)\}$$

$$\frac{\partial^2 U}{\partial x^2} = \mathrm{Re}\{2e^{(u+vi)^2}(u_y u_x + uu_{xy} + u_y v_x i + uv_{xy}i + u_{xy}vi + u_x v_y i - v_y v_x - vv_{xy})$$

$$+ (uu_x + uv_x i + u_x vi - vv_x)(4e^{(u+vi)})(u+vi)(u_x + v_x i)\}$$

$$\frac{\partial U}{\partial y} = \mathrm{Re}\{e^{(u+vi)^2}2(u+iv)(u_y+v_y i)\} = \mathrm{Re}\{2e^{(u+vi)^2}(uu_y + uv_y i + u_y vi - vv_y)$$

$$\frac{\partial^2 U}{\partial y^2} = \mathrm{Re}\{2e^{(u+vi)}(uu_{yy} + u_y u_y + uv_{yy}i + u_y v_y i + u_{yy}vi + u_y v_y i - v_y v_y - vv_{yy})$$

$$+ (uu_y + uv_y i + u_y vi - vv_y)4e^{(u+vi)}(u+vi)(u_y + v_y i)\}$$

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \overset{?}{=} 0$$

$\boxed{\text{Yes}}$ ✓

Problem (6) (20 Points) Consider two functions:

$$P(x,y) = x^3 - 3xy^2 + 2x$$
$$R(x,y) = x^3 - 3x^2y - 2y$$

Question: Could $P(x,y)$ and/or $R(x,y)$ conceivably be the real part of an (everywhere) analytic function? Provide complete justification and circle the appropriate answer at the bottom of the page.

[No justification $\Longrightarrow$ not much credit.] In (possible) affirmative cases, it is *not* requjired that you produce the harmonic conjugate.

suppose $\psi = P + Yi$

$$\frac{\partial P}{\partial x} = 3x^2 - 3y^2 + 2$$
$$\frac{\partial P}{\partial y} = -6xy$$

$$\frac{\partial Y}{\partial x} = -\frac{\partial P}{\partial y} = 6xy$$

$$Y = \int 6xy\,dx = 3x^2y + g(y)$$

$$\frac{\partial Y}{\partial y} = 3x^2 + g'(y) = \frac{\partial P}{\partial x} = 3x^2 - 3y^2 + 2$$

$$\Rightarrow g'(y) = -3y^2 + 2$$
$$g(y) = \int -3y^2 + 2\,dy = -y^3 + 2y$$

$$\frac{\partial Y}{\partial x} = \frac{\partial}{\partial x}(3x^2y - y^3 + 2y) = 6xy$$

So, $\frac{\partial Y}{\partial x} = -\frac{\partial P}{\partial y}$ and $\frac{\partial Y}{\partial y} = \frac{\partial P}{\partial x}$

P could be $\mathrm{Re}\{\psi\}$.

ok

✓

Suppose $\psi = R + Yi$

$$\frac{\partial R}{\partial x} = 3x^2 - 6xy$$
$$\frac{\partial R}{\partial y} = -3x^2 - 2$$

$$\frac{\partial Y}{\partial x} = -\frac{\partial R}{\partial y} = 3x^2 + 2$$

$$Y = \int 3x^2 + 2\,dx = x^3 + 2x + g(y)$$

$$\frac{\partial Y}{\partial y} = g'(y) = \frac{\partial R}{\partial x} = 3x^2 - 6xy$$

This ~~means a~~ function of y has x terms in it, this is a contradiction.

R could not be part of an everywhere-analytic function

P only R only Both Neither (Circle one)