(1)

$$P(z) = P(0) + z\,Q(z)$$

$$\Rightarrow \quad \frac{1}{z} \underset{\substack{\uparrow \\ P(z) \text{ has no zeros}}}{=} \frac{P(z)}{zP(z)} = \frac{P(0)}{zP(z)} + \frac{Q(z)}{P(z)}$$

$$\Rightarrow \quad 2\pi i = \oint_{|z|=R} \frac{1}{z}\,dz = \oint_{|z|=R} \frac{P(0)}{zP(z)}\,dz + \underbrace{\oint_{|z|=R} \frac{Q(z)}{P(z)}\,dz}_{\substack{\| \\ 0}}$$

$\leftarrow$ Cauchy's Theorem (since $P(z)$ has no zeros $\Rightarrow \frac{Q(z)}{P(z)}$ analytic on $|z|<R$)

$$\Rightarrow \quad 2\pi = |2\pi i| = \left| \oint_{|z|=R} \frac{P(0)}{zP(z)}\,dz \right| \underset{\substack{\uparrow \\ z=Re^{i\theta} \\ dz = Rie^{i\theta}d\theta}}{=} \left| \int_0^{2\pi} \frac{P(0)}{\cancel{Re^{i\theta}}P(Re^{i\theta})} \cdot \cancel{R}i\cancel{e^{i\theta}}\,d\theta \right|$$

$$\leq |P(0)| \cdot 2\pi \cdot \max_{|z|=R} \frac{1}{|P(z)|} \xrightarrow[\text{as } R\to\infty]{} 0$$

since if $P(z)$ has degree $n \geq 1$, (non-constant)

then $\left|\frac{P(z)}{z^n}\right| = \left|\frac{a_n z^n + \cdots}{z^n}\right| \xrightarrow[\text{as } z\to\infty]{} |a_n| \neq 0$

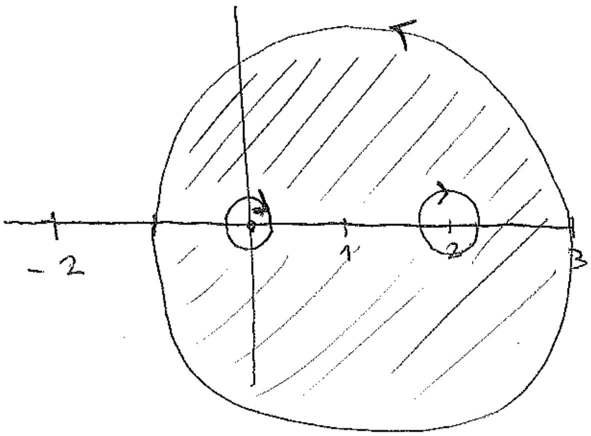
so $|P(z)| \longrightarrow \infty$ as $|z| \to \infty$

$\therefore \quad 2\pi \leq 0 \quad !!$

(2)

$$\oint_{|z-1|=2} \frac{dz}{z^2(z^2-4)\,e^z} \underset{\substack{\uparrow\\ \text{Cauchy's thm.}}}{=} \underbrace{\oint_{|z|=\varepsilon} \frac{dz}{z^2(z^2-4)\,e^z}} + \underbrace{\oint_{|z-2|=\varepsilon} \frac{dz}{z^2(z^2-4)\,e^z}}$$

Cauchy's integral thm.



$$\oint_{|z|=\varepsilon} \frac{dz}{z^2(z^2-4)\,e^z} = \frac{2\pi i}{1!}\,\frac{d}{dz}\left(\frac{1}{(z^2-4)e^z}\right)\bigg|_{z=0} = 2\pi i\left(-\frac{2ze^z+(z^2-4)e^z}{(z^2-4)^2\,e^{2z}}\right)\bigg|_{z=0} = 2\pi i\cdot\frac{4}{16}$$

$$\oint_{|z-2|=\varepsilon} \frac{dz}{z^2(z^2-4)\,e^z} = 2\pi i \oint \frac{1}{z^2(z+2)\,e^z}\bigg|_{z=2} = 2\pi i\,\frac{1}{16e^2}$$

$$= \frac{\pi i}{2} + \frac{\pi i}{8e^2}\,.$$

③

(a) Ratio test

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)/n!}{\alpha(\alpha-1)\cdots(\alpha-n)/(n+1)!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n+1}{\alpha-n}\right|$$

$$= 1.$$

$$\therefore \boxed{R=1}.$$

(b)

$$f(z) = (1+z)^{\alpha} = e^{\alpha \operatorname{Log}(1+z)}$$

$$\Rightarrow f'(z) = \alpha\cdot\frac{1}{1+z}\cdot e^{\alpha\operatorname{Log}(1+z)} = \alpha(1+z)^{\alpha-1} \quad ; \quad f'(0) = \alpha$$

$$f''(z) = \alpha\cdot(\alpha-1)(1+z)^{\alpha-2} \quad ; \quad f''(0) = \alpha(\alpha-1)$$

$$\vdots$$

$$f^{(n)}(z) = \alpha\cdot(\alpha-1)\cdots(\alpha-n+1)(1+z)^{\alpha-n} \quad ; \quad f^{(n)}(0) = \alpha(\alpha-1)\cdots(\alpha-n+1)$$

$$\therefore f(z) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n$$

$$= \sum_{n=0}^{\infty}\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}z^n = \sum_{n=0}\binom{\alpha}{n}z^n.$$

④

$$f(z) = \frac{1}{(z-1)(z-2)}$$

Regions of analyticity : $\{0 < |z-1| < 1\}$ and $\{|z-1| > 1\}$.

- $\boxed{\text{On } \{0 < |z-1| < 1\}}$:

$$\frac{1}{z-2} \underset{\substack{\uparrow \\ \text{analytic} \\ \text{for } |z-1|<1}}{=} \frac{1}{z-1-1} = -\frac{1}{1-(z-1)} = -\sum_{k=0}^{\infty} (z-1)^k$$

$$\Rightarrow \boxed{\frac{1}{(z-1)(z-2)} = -\sum_{k=-1}^{\infty} (z-1)^k}$$

- $\boxed{\text{On } \{|z-1| > 1\}}$:

$$\frac{1}{z-2} \underset{\substack{\uparrow \\ \text{analytic} \\ \text{for } |z-1|>1 \\ (\Leftrightarrow \left|\frac{1}{z-1}\right| < 1)}}{=} \frac{1}{z-1-1} = \frac{1}{z-1} \cdot \frac{1}{\left(1-\frac{1}{z-1}\right)} = \frac{1}{z-1} \sum_{k=0}^{\infty} \frac{1}{(z-1)^k}$$

$$\Rightarrow \boxed{\frac{1}{(z-1)(z-2)} = \sum_{k=2}^{\infty} \frac{1}{(z-1)^k}}$$