(1)

$$e^{z+\lambda} = e^{z}$$

$z = x+iy$

$\lambda = \alpha + i\beta$

$$\Rightarrow \underbrace{e^{(x+iy)+(\alpha+i\beta)}}_{\substack{\| \\ e^{(x+\alpha)+i(y+\beta)} \\ \| \\ e^{x+\alpha}\cdot e^{i(y+\beta)}}} = \underbrace{e^{x+iy}}_{\substack{\| \\ e^{x}\cdot e^{iy}}}$$

$$\Rightarrow \left.\begin{array}{l} e^{x+\alpha} = e^{x} \\ e^{i(y+\beta)} = e^{iy} \end{array}\right\} \begin{array}{l} \overset{x,\alpha\in\mathbb{R}}{\Rightarrow} \quad \boxed{\alpha = 0} \\ \Rightarrow \quad \cos(y+\beta) + i\sin(y+\beta) = \cos y + i \sin y \end{array}$$

$$\Rightarrow \left.\begin{array}{l} \cos(y+\beta) = \cos y \\ \sin(y+\beta) = \sin y \end{array}\right\}$$

$$\Rightarrow \beta = 2\pi i n \quad (n\in\mathbb{Z})$$

$$\therefore \boxed{\lambda = \alpha + i\beta = 2\pi i n \quad (n \in \mathbb{Z})}.$$

(2)

$$f(z) = z^3 = (x+iy)^3$$

$$= x^3 + 3x^2 iy + 3x(-y^2) + (-i)y^3$$

$$= \underbrace{(x^3 - 3xy^2)}_{u(x,y)} + i\underbrace{(3x^2y - y^3)}_{v(x,y)}$$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= 3x^2 - 3y^2 \\ \frac{\partial v}{\partial y} &= 3x^2 - 3y^2 \end{aligned}\right\} \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\left.\begin{aligned} \frac{\partial u}{\partial y} &= -6y \\ \frac{\partial v}{\partial x} &= 6y \end{aligned}\right\} \Rightarrow \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} .$$

(3)

(a) $\cdot$ $$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} = \cos\theta\,\frac{\partial u}{\partial x} + \sin\theta\,\frac{\partial u}{\partial y}$$

$\cdot$ $$\frac{\partial^2 u}{\partial r^2} = \cos\theta\left(\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial \theta} + \frac{\partial^2 u}{\partial x\partial y}\frac{\partial y}{\partial \theta}\right) + \sin\theta\left(\frac{\partial^2 u}{\partial y\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial \theta}\right)$$

$$= \cos^2\theta\,\frac{\partial^2 u}{\partial x^2} + 2\sin\theta\cos\theta\,\frac{\partial^2 u}{\partial x\partial y} + \sin^2\theta\,\frac{\partial^2 u}{\partial y^2}$$

$\cdot$ $$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\,\frac{\partial u}{\partial x} + r\cos\theta\,\frac{\partial u}{\partial y}$$

$\cdot$ $$\frac{\partial^2 u}{\partial \theta^2} = -r\cos\theta\,\frac{\partial u}{\partial x} + (-r\sin\theta)\left[\frac{\partial^2 u}{\partial x^2}(-r\sin\theta) + \frac{\partial^2 u}{\partial x\partial y}\,r\cos\theta\right] - r\sin\theta\,\frac{\partial u}{\partial y}$$

$$+ r\cos\theta\left[\frac{\partial^2 u}{\partial x\partial y}(-r\sin\theta) + \frac{\partial^2 u}{\partial y^2}\,r\cos\theta\right]$$

$$\Rightarrow \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \cos^2\theta\,\frac{\partial^2 u}{\partial x^2} + 2\sin\theta\cos\theta\,\frac{\partial^2 u}{\partial x\partial y} + \sin^2\theta\,\frac{\partial^2 u}{\partial y^2}$$

$$+ \frac{1}{r}\cos\theta\,\frac{\partial u}{\partial x} + \frac{1}{r}\sin\theta\,\frac{\partial u}{\partial y}$$

$$-\frac{1}{r}\cos\theta\,\frac{\partial u}{\partial x} + \sin^2\theta\,\frac{\partial^2 u}{\partial x^2} - \sin\theta\cos\theta\,\frac{\partial^2 u}{\partial x\partial y} - \frac{1}{r}\sin\theta\,\frac{\partial u}{\partial y}$$

$$-\sin\theta\cos\theta\,\frac{\partial^2 u}{\partial x\partial y} + \cos^2\theta\,\frac{\partial^2 u}{\partial y^2}$$

$$= \underbrace{(\cos^2\theta + \sin^2\theta)}_{=1}\frac{\partial^2 u}{\partial x^2} + \underbrace{(\sin^2\theta + \cos^2\theta)}_{=1}\frac{\partial^2 u}{\partial y^2}.$$

(b) $$\log|\overset{re^{i\theta}}{\overset{\shortparallel}{z}}| = \log r \Rightarrow u = \log r,\ (v = 0)$$

$$\Rightarrow \frac{\partial u}{\partial r} = \frac{1}{r}\ ;\quad \frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}\ ;\quad \frac{\partial u}{\partial \theta} = 0\ ;\quad \frac{\partial^2 u}{\partial \theta^2} = 0$$

$$\Longrightarrow \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{r^2} + \frac{1}{r}\cdot\frac{1}{r} + 0 = 0 \Rightarrow$$ $\log|z|$ harmonic on $\mathbb{C}\setminus\{0\}$.

④

$$\omega = \frac{-y\,dx + x\,dy}{x^2+y^2} \qquad ((x,y) \neq (0,0))$$

$$\Rightarrow \quad P = \frac{-y}{x^2+y^2} \qquad Q = \frac{x}{x^2+y^2}$$

(a)

$$\left.\begin{aligned} \frac{\partial P}{\partial y} &= \frac{-(x^2+y^2) + y\cdot 2y}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2} \\ \frac{\partial Q}{\partial x} &= \frac{(x^2+y^2) - x\cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \end{aligned}\right)\!\!\Bigg) \Rightarrow \quad \omega \text{ is closed.}$$

(b) ~~Since integral of path [illegible] exact [illegible]~~

It suffices to show that $\displaystyle\int_\gamma \omega \neq 0$ for some <u>closed</u> path γ inside the annulus. A.

Let $r>0$ be such that $\gamma(\theta) = (r\cos\theta, r\sin\theta)$ is inside A.
$(0 \leq \theta \leq 2\pi)$

Then

$$\int_\gamma \omega = \int_0^{2\pi} \frac{-r\sin\theta}{r^2}\frac{dx}{d\theta}\,d\theta + \int_0^{2\pi} \frac{r\cos\theta}{r^2}\frac{dy}{d\theta}\,d\theta$$

$$= \int_0^{2\pi} -\frac{\sin\theta}{\cancel{r}}\cdot(-\cancel{r}\sin\theta)\,d\theta + \int_0^{2\pi} \frac{\cos\theta}{\cancel{r}}\,\cancel{r}\cos\theta\,d\theta$$

$$= \int_0^{2\pi} \underbrace{\sin^2\theta + \cos^2\theta}_{=1}\,d\theta = 2\pi \neq 0 \qquad \blacksquare$$