Name: ____________

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MATH 131A - MIDTERM EXAM 2

0.1. **Instructions.** This is a 50 minute exam. You should feel free to quote any theorems proved in class, as well as anything proved in the homework or discussion section. There are 6 questions—you are required to do the final true/false question, and choose 4 of the remaining 5. Only 4 problems other than the true/false question will be graded so *you should indicate which problems you want graded*, in the case that you attempt all 6. Each question is worth 10 points.

Exercise 0.1. Suppose $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ are sequences. Show

$$\limsup s_n t_n \leq \limsup s_n \limsup t_n.$$

Proof. Given any N, if $n > N$ then $a_n \leq \sup\{a_m : m > N\}$ and $b_n \leq \sup\{b_m : m > N\}$ so $a_n b_n \leq \sup\{a_m : m > N\}\sup\{b_m : m > N\}$. As $n > N$ is arbitrary, we have

$$\sup\{a_m b_m : m > N\} \leq \sup\{a_m : m > N\}\sup\{b_m : m > N\}.$$

Taking the limit as N tends to ∞ yields the desired inequality. □

Exercise 0.2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Show that f is not continuous at any point in the domain.

Proof. Fix $x_0 \in \mathbb{R}$. Case 1: $x_0 \in \mathbb{Q}$. Then for any $\delta > 0$, we can find some irrational x with $|x - x_0| < \delta$ by the density of the irrationals so $|f(x) - f(x_0)| = |0 - 1| = 1$, which shows f is not continuous at x_0. Case 2: $x_0 \notin \mathbb{Q}$. Then, for any $\delta > 0$, there is some rational x so that $|x - x_0| < \delta$ and then $|f(x) - f(x_0)| = |1 - 0| = 1$. It follows that for any $x_0 \in \mathbb{R}$, there is an ϵ, namely 1, so that for any $\delta > 0$, there is x with $|x - x_0| < \delta$ and $|f(x) - f(x_0)| \geq 1$, so f is not continuous at x_0. □

Exercise 0.3. Let $(s_n)_{n\in\mathbb{N}}$ be a sequence defined by s_n is the nth number after the decimal point in π. Since $\pi = 3.14159...$ we have $s_1 = 1$, $s_2 = 4$, $s_3 = 1$, and so on. Prove that $(s_n)_{n\in\mathbb{N}}$ has a convergent subsequence.

Proof. By the peak point lemma, $(s_n)_{n\in\mathbb{N}}$ has a monotone subsequence. It is bounded (because contained in $[0, 10]$) so converges. □

Exercise 0.4. Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function and for each $\epsilon > 0$, there is an $x \in [a, b]$ so that $|f(x) - l| < \epsilon$. Prove that there is some $x_0 \in [a, b]$ so that $f(x_0) = l$.

Hint: Find a sequence to which you can apply the Bolzano-Weierstrass theorem.

Proof. For each $n \in \mathbb{N}$, pick x_n so that $|f(x_n) - l| < \frac{1}{n}$. Let $(x_{n_k})_{k\in\mathbb{N}}$ be a convergent subsequence and let $x_0 = \lim_{k\to\infty} x_{n_k}$. We have $x_0 \in [a, b]$, because $[a, b]$ is a closed interval, and, by continuity, $(f(x_{n_k}))_{k\in\mathbb{N}}$ converges to $f(x_0)$. Given any $\epsilon > 0$, choose N so that both $\frac{1}{N} < \frac{\epsilon}{2}$ and $n \geq N$ implies $|f(x_{n_k}) - f(x_0)| < \frac{\epsilon}{2}$. Then for any $n \geq N$,

$$|f(x_0) - l| = |f(x_0) - f(x_{n_k}) + f(x_{n_k}) - l| \leq |f(x_0) - f(x_{n_k})| + |f(x_{n_k}) - l| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This shows $|f(x_0) - l| < \epsilon$ for arbitrary $\epsilon > 0$ so $f(x_0) - l = 0$, i.e. $f(x_0) = l$. □

Exercise 0.5. Suppose f is a uniformly continuous function and $(s_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the domain of f. Prove that $(f(s_n))_{n\in\mathbb{N}}$ is a Cauchy sequence.

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Proof. Fix $\epsilon > 0$. There is a $\delta > 0$ so that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. There is N so that if $n, n' \geq N$, then $|s_n - s_{n'}| < \delta$ and hence $|f(s_n) - f(s_{n'})| < \epsilon$. As $\epsilon > 0$ was arbitrary, this shows $(f(s_n))_{n\in\mathbb{N}}$ is Cauchy. □

Exercise 0.6. Indicate whether the following statements are true or false:

(1) If $f : [a, b] \to \mathbb{R}$ is an unbounded function, then f is not continuous.
(2) If $(s_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the open interval (a, b), then it converges to an element of $[a, b]$.
(3) For any sequence $(s_n)_{n\in\mathbb{N}}$, $\limsup s_n$ is defined.
(4) If $f : [1, 2] \to \mathbb{R}$ is the function $f(x) = \frac{1}{x^2}$, then f is uniformly continuous.
(5) If $f : [a, b] \to \mathbb{R}$ and $g : [b, c] \to \mathbb{R}$ are continuous functions and $f(b) = g(b)$, then $h : [a, c] \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} f(x) & x \in [a, b] \\ g(x) & x \in [b, c] \end{cases}$$

is continuous.

Proof. (1). True - boundedness states that a continuous function on a closed interval is bounded.

(2). True. Every convergent sequence in $[a, b]$ converges to a point in $[a, b]$ and every Cauchy sequence converges. As $(s_n)_{n\in\mathbb{N}}$ is in (a, b), it is, in particular, in $[a, b]$.

(3). True. Either $\limsup s_n = \infty$ or the sequence is bounded and the sequence $(t_N)_{N\in\mathbb{N}}$ defined by $t_N = \sup\{s_n : n > N\}$ is decreasing and therefore has a defined limit (possibly $-\infty$).

(4). True, as it is a continuous function whose domain is a closed interval.

(5). True. On the homework, we checked that h continuous if and only if given any monotonic sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathrm{dom}(h)$ that converges to $x_0 \in \mathrm{dom}(h)$, then $(h(x_n))_{n\in\mathbb{N}}$ converges to $h(x_0)$. Given any monotonic sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathrm{dom}(h)$ there is some N so that either $x_n \in [a, b]$ for all $n \geq N$ or $x_n \in [b, c]$ for all $n \geq N$ and hence the desired convergence follows from the continuity of f and g. (Note: this can also be proved directly using the $\delta - \epsilon$ definition). □