Name: ____________

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MATH 131A - MIDTERM EXAM

0.1. **Instructions.** This is a 50 minute exam. You should feel free to quote any theorems proved in class, as well as anything proved in the homework or discussion section, unless specifically instructed otherwise. There are 6 questions—you are required to do the first true/false question, and choose 4 of the remaining 5. Only 4 problems other than the true/false question will be graded so *you should indicate which problem you do not want graded by marking it with a large X across the page.* Each question is worth 10 points.

The axioms of an ordered field

(1) **O1**: Given a and b, either $a \leq b$ or $b \leq a$.
(2) **O2**: If $a \leq b$ and $b \leq a$, then $a = b$.
(3) **O3**: If $a \leq b$ and $b \leq c$, then $a \leq c$.
(4) **O4**: If $a \leq b$, then $a + c \leq b + c$.
(5) **O5**: If $a \leq b$ and $0 \leq c$, then $ac \leq bc$.

Exercise 0.1. Indicate whether the following statements are true or false:

(1) The maximum of a bounded subset of $\mathbb{Q}$, if it exists, is rational.
(2) Suppose $S \subseteq T$ are non-empty. Then $\inf(T) \leq \inf(S) \leq \sup(S) \leq \sup(T)$.
(3) The sequence $(s_n)_{n\in\mathbb{N}}$ converges if there is some s so that the sequence $(s_n - s)_{n\in\mathbb{N}}$ converges to 0.
(4) If $S, T \subseteq \mathbb{R}$ have non-empty intersection, then $\sup(S \cap T) = \min(\sup(S), \sup(T))$.
(5) Suppose $(s_n)_{n\in\mathbb{N}}$ is a sequence which does not converge. Then the sequence of absolute values $(|s_n|)_{n\in\mathbb{N}}$ also does not converge.

Proof. (1) True, the maximum of a set, if it exists, is an element of the set. So if the maximum of a set of rationals exists, it is rational.

(2) True, every lower bound for $\inf(T) \leq t$ for every element $t \in T$ so, since $S \subseteq T$, $\inf(T) \leq s$ for every $s \in S$. This means that $\inf(T)$ is a lower bound for S so is therefore less than or equal to the greater lower bound of S. This shows $\inf(T) \leq \inf(S)$. The inequality $\sup(S) \leq \sup(T)$ is similar and $\inf(S) \leq \sup(S)$ is clear.

(3) True. This is clear from the definitions, since $|s_n - s| = |(s_n - s) - 0|$. In more details, suppose $(s_n)_{n\in\mathbb{N}}$ converges to s. Then for all $\epsilon > 0$, there is N so that $n \geq N$ implies $|s_n - s| < \epsilon$. In other words, for all $n \geq N$, $|(s_n - s) - 0| < \epsilon$ so $(s_n - s)_{n\in\mathbb{N}}$ converges to 0. On the other hand, $(s_n - s)_{n\in\mathbb{N}}$ converges to 0, then for any $\epsilon > 0$, there is N so that $n \geq N$ implies $|(s_n - s) - 0| < \epsilon$, hence $|s_n - s| < \epsilon$. This shows $(s_n)_{n\in\mathbb{N}}$ converges to 0.

(4) False. Take $S = [0, 1) \cup [2, 3)$ and take $T = [0, 2)$. Then $S \cap T = [0, 1)$. So $\sup(S) = 3$, $\sup(T) = 2$, and $\sup(S \cap T) = 1$.

(5) False. Take $s_n = (-1)^n$. We know this does not converge and yet $|s_n| = 1$ for all $n \in \mathbb{N}$ which clearly converges. □

Exercise 0.2. Prove that

$$2^{n+3} \geq (n + 3)^2$$

for all natural numbers n.

Hint: $(k + 4)^2 = ((k + 3) + 1)^2 = (k + 3)^2 + 2(k + 3) + 1$.

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Proof. The base case $n = 1$ is the statement that $2^4 \leq 4^2$ which is true because both sides are 16.

Suppose we know that the statement is true for k. Then have

$$\begin{aligned}(k+4)^2 &= (k+3)^2 + 2(k+3) + 1 \\ &\leq 2^{k+3} + 2(k+3) + 1.\end{aligned}$$

Observe that $2(k+3)+1 \leq 2^{k+3}$ for all k: at $k = 1$, this says $9 \leq 16$. If its true for k, then we have $2^{k+4} \geq 2(2(k+3)+1) = (k+3)+(k+3)+2 \geq (k+4)+1$, which proves the induction step. So we have

$$2^{k+3} + 2(k+3) + 1 \leq 2^{k+3} + 2^{k+3} = 2^{k+4}.$$

Putting it all together, we have

$$(k+4)^2 \leq 2^{k+4}.$$

□

Exercise 0.3. Find all rational roots of the polynomial $x^2 + \frac{1}{2}x + 1$ or prove there are none.

Proof. Note that the given polynomial does not have integer coefficients so the rational roots theorem does not directly apply. But multiplying by 2, we have $x^2 + \frac{1}{2}x + 1 = 0$ if and only if $2x^2 + x + 2 = 0$, and the latter polynomial does have only integral coefficients. If $\frac{c}{d}$ is a rational solution with $\gcd(c, d) = 1$, then c, d divide 2, so $c = \pm 1, 2$ and $d = \pm 2$ so $\frac{c}{d} = \pm\frac{1}{2}, \pm 1, \pm 2$. Plugging in, we see that none of these roots are solutions, so by the rational roots theorem the polynomial has no rational roots. □

Exercise 0.4. Prove from the axioms of an ordered field (without citing any results from class) that, in any ordered field F, for all $a, b, c, d \in F$, if $a \leq b$ and $c \leq d$, then $a+c \leq b+d$.

Proof. Suppose $a \leq b$ and $c \leq d$. By O4, because $a \leq b$, we have $a + c \leq b + c$. Also because $c \leq d$, by O4 again, we have $b + c \leq b + d$. Since we have $a + c \leq b + c$ and $b + c \leq b + d$, by O3, we have $a + c \leq b + d$. □

Exercise 0.5. Prove the following limit law: if $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ are sequences that converge to s and t respectively, then $(s_n + t_n)_{n\in\mathbb{N}}$ converges to $s + t$.

Proof. Fix $\epsilon > 0$. As $(s_n)_{n\in\mathbb{N}}$ converges to s, there is N_0 so that, if $n \geq N_0$, then $|s_n - s| < \epsilon/2$. Likewise, as $(t_n)_{n\in\mathbb{N}}$ converges to t, there is N_1 so that, if $n \geq N_1$, then $|t_n - t| < \epsilon/2$. Set $N = \max\{N_0, N_1\}$. Then for all $n \geq N$, we have

$$|(s_n + t_n) - (s + t)| = |(s_n - s) + (t_n - t)| \leq |s_n - s| + |t_n - t| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

As ϵ is arbitrary, this shows $(s_n + t_n)_{n\in\mathbb{N}}$ converges to $s + t$. □

Exercise 0.6. Suppose A and B are non-empty sets of positive real numbers and assume A and B are bounded above. Let $C = \{ab : a \in A, b \in B\}$. Prove the following:

(1) C is bounded above.
(2) $\sup(C) = \sup(A)\sup(B)$.

Proof. Because A is bounded above and consists only of positive elements, there is a positive $M_0 = \sup(A)$ so that $a \leq M_0$ for every $a \in A$. Likewise, because B is bounded above and consists only of positive elements, there is some positive $M_1 = \sup(B)$ so that $b \leq M_1$ for all $b \in B$. Since every $b \in B$ is positive, we have $ab \leq bM_0$ for every $a \in A$ and $b \in B$. Likewise, because M_0 is positive, we have $bM_0 \leq M_0M_1$ for every $b \in B$. Putting it together, we get $ab \leq M_0M_1$ for every $a \in A$ and $b \in B$ so C is bounded above by M_0M_1. Since $M_0M_1 = \sup(A)\sup(B)$ is an upper bound for C, the least upper bound for C satisfies $\sup(C) \leq \sup(A)\sup(B)$.

To complete the argument, we have to show $\sup(C) \geq \sup(A)\sup(B)$. Towards contradiction, suppose $\sup(C) < \sup(A)\sup(B)$. Given any small $\epsilon > 0$ with $\sup(A)\sup(B) - \cup(C) > \epsilon$. Then $\sup(A) - \epsilon/2\sup(B) < \sup(A)$ so there is $a \in A$ with $\sup(A) - a < \epsilon/2\sup(B)$ and likewise $\sup(B) - \epsilon/2\sup(A) < \sup(B)$ so there is $b \in B$ with $\sup(B) - b < \epsilon/2\sup(A)$. As $a \in A$ and $b \in B$, we know $ab \in C$ hence $ab \leq \sup(C)$. Therefore $\sup(A)\sup(B) - \sup(C) \leq \sup(A)\sup(B) - ab$. Additionally,

$$\begin{aligned}
\sup(A)\sup(B) - ab &= \sup(A)\sup(B) - \sup(A)b + \sup(A)b - ab \\
&= \sup(A)(\sup(B) - b) + (\sup(A) - a)b \\
&< \frac{\sup(A)\epsilon}{2\sup(A)} + \frac{\sup(B)\epsilon}{2\sup(B)} \\
&= \epsilon.
\end{aligned}$$

This shows that $\epsilon < \sup(A)\sup(B) - \sup(C) \leq \sup(A)\sup(B) - ab < \epsilon$, which is impossible. This contradiction completes the proof. □