Fall 2015 MATH 131AH Midterm

Please write clearly, and show your reasoning with mathematical rigor.

Name:

Problem	Score
1	10
2	9
3	8
4	8
5	0
Total	35

1. (10pt) 10

(a) State the least upper bound property for the real number system $\mathbb{R}$.

(b) Using (a), show that the set $\{n^2 : n \in \mathbb{N}\}$ has no upper bound in $\mathbb{R}$.

$\overset{\shortparallel}{E}$

a) If $E \subseteq \mathbb{R}$, $E \neq \emptyset$, and E is bounded above, $\exists \sup E \in \mathbb{R}$.

b) Suppose not. Then $\exists$ upper bound of E in $\mathbb{R}$.

Then by a), $\exists \sup E. = \alpha$.

But $\exists n^2 \mid \alpha - \epsilon < n^2 \leq \alpha$.

Arbitrarily Chosen

Choose $\epsilon = 2n + 1 \longrightarrow \epsilon$ cannot be smaller than 1, but $d(n_1, n_2) \geq 1$ unless $n_1 = n_2$. $n \in \mathbb{N}$.

$\Rightarrow \alpha < n^2 + 2n + 1$

$\Rightarrow \alpha < (n+1)^2$ ✗

b/c $(n+1)^2 \in E$.

By the contrapositive, $\{n^2 : n \in \mathbb{N}\}$ has no upper bound in E. ∎

* Discrete Metric is open.
↳ not perfect, but still open.

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2. (15pt) (True or False) If true, you do not need to prove it. If false, please provide a counterexample: you do not need to prove that it is a counterexample.

(a) Countable intersections of open sets are always open. False.

(b) Countable intersection of compact sets in $\mathbb{R}^k$ are compact. True. Corrections.

(c) Any non-empty open set in a metric space is uncountable. ~~True.~~ False

(d) Cauchy sequences in a metric space are convergent sequences. ~~True.~~ False

(e) $\liminf s_n < +\infty$ for any sequence s_n in $\mathbb{R}$. False.

a) Counter-Ex. Suppose $A_n = \{(-\frac{1}{n}, \frac{1}{n})\}$.

$\bigcap_{n=1}^{\infty} A_n = \{0\}$ which is closed and not open,

Open. Intervals A_n are countable, by $\varphi : \mathbb{N} \to A_n$. ∎

b) Compact ⇒ Closed and Bounded.

$\bigcap_n^{\infty} K$ is closed, ⇒ $\bigcap_n K$ is compact in $\mathbb{R}^k$.

(Heine-Borel)

c) Choose $\mathbb{Q} \subseteq \mathbb{Q}$. $\mathbb{Q}$ is open in $\mathbb{Q}$ and countable. ∎

d) $A_n = \{\sqrt{2} - \frac{1}{n}\}$ $\lim_{n\to\infty} A_n = \sqrt{2} \notin \mathbb{Q}$. ∎

e) Suppose $s_n = \{n \mid n \in \mathbb{N}\}$.

Then $\liminf s_n = \limsup s_n = \infty$. ∎

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3. (10pt) Let B be a nonempty subset of a metric space (X, d). Show that if $x \in \bar{B}$ then there is a sequence x_n in B which converges to x.

① Suppose $x \in B'$.

Then $\forall \epsilon > 0$, pick $x_1, x_2, x_3, \ldots$

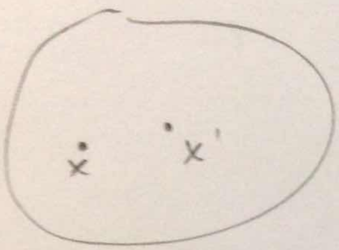
s.t. $x_1 \in N(x, \epsilon_1)$, $x_2 \in N(x, \epsilon_2)$ ...

for arbitrarily small ϵ.

$\epsilon_1 > \epsilon_2 > \epsilon_3 \ldots$

Then $x_n = \{x_1, x_2, x_3 \ldots\}$

converges to x. ■



At least need $\varepsilon_n \to 0$

(-2)

② Suppose $x \in B$. ~~Suppose not.~~ Not necessary ✓

But $\exists$ a finite sequence $x_1 = x_2 = \cdots = x$

in B which converges to x. ✱.

$\Rightarrow \exists\, x_n \to x \in B$. ■

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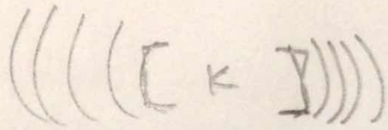
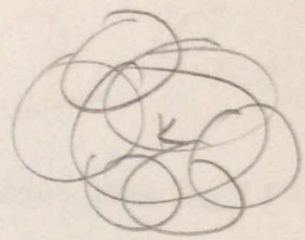




4. (10pt)

(a) State the definition of a compact set.

(b) Using only the definition of a compact set, prove that every compact set is bounded.

a) K is compact if every open cover of K contains a finite subcover of K.

b) Suppose K is compact. Then for all $\{G_\alpha\}$: open cover of K, $\exists \bigcup_\alpha^n G_\alpha$ a finite subcover. ?

Then take the diameter of the finite subcover

$\operatorname{diam} \bigcup_\alpha^n G_\alpha = \sup d(p,q) \quad \forall p,q \in \bigcup_\alpha^n G_\alpha$.

G_α may not be bounded.

K is then bounded by

$N(p, \operatorname{diam} \bigcup_\alpha^n G_\alpha)$ because

$K \subseteq \bigcup_\alpha^n G_\alpha$.

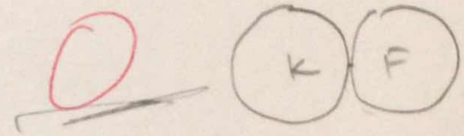
??

$d(p,k) < \operatorname{diam} \bigcup_\alpha^n G_\alpha$

for all $k \in K$.

(−2)

~~Study ϵ/N proofs~~



5. (10pt) Let $K, F \subset \mathbb{R}^2$, and let K be compact and F be closed. Show that if F is disjoint with K then

$$\inf_{x \in K, y \in F} d(x, y) > 0.$$

(Hint: consider sequences with $d(x_n, y_n) \to 0$.)

$F \cap K = \phi$ $\quad$ K compact

F closed

???

~~Pf. Suppose not. $\inf_{x \in K, y \in F} d(x,y) = 0$~~

~~$\Rightarrow$ But $\Rightarrow$ $x = y$ $\Rightarrow$ $F \cap K \neq \phi$.~~

~~So x is either in K or F.~~

~~Suppose $x \in K$.~~

~~Then $\exists\, x_n \to x$ because K is compact.~~

Suppose not. Then $\inf_{x \in K, y \in F} d(x,y) = 0$

$\forall n \in \mathbb{N}$ So $\exists$ sequence $x_n \in K$, $y_n \in F$ s.t.

$d(x_n, y_n) < \frac{1}{n}$

Then $d(x_n, y_n) \to 0$. $\quad \{x_n\}$: sequence in K

Then $\exists$ subseq x_{n_k} that converges to $x \in K$ (by compactness.).

$x \in F'$ since $d(x, y_n) \leq \underset{\downarrow\, 0}{d(x, x_n)} + d(x_n, y_n) \to 0$

$\Rightarrow x \in F$. $\divideontimes$ ■