Fall 2011 MATH 131AH Midterm

1. (10pt) Let x be a real positive number. Show that $x^{m+n} = x^m x^n$ for any integers m and n.

Proof by Induction.

Base. $m = 0 \Rightarrow X^{0+n} = X^n = X^0 \cdot X^n$ $\blacksquare$

Inductive Step. $m = m+1$ $(m > 0)$

$X^1 = X$

$\Rightarrow X^{(m+1)+n} = X^{m+(1+n)} = X^{m+(n+1)} = X^m \cdot X^n \cdot X$

$\Rightarrow X^m \cdot X \cdot X^n = X^{m+1} X^n$ $\blacksquare$

Inductive Step. $m < 0 \Rightarrow -m > 0$ Show. $m = m-1$

$\Rightarrow$ by previous argument $-m = 1-m$

$X^{(-m+1)+n} \Rightarrow X^{-m+1} x^n$

done $\blacksquare$

2. (10pt) (True or False) If true, you do not need to prove it. If false, please provide a counterexample: you do not need to prove that it is a counterexample.

(a) If a metric space is compact then it is also complete.

Compact $\Rightarrow$ Complete

Complete $\not\Rightarrow$ Compact

True.

(b) If a metric space is closed and bounded then it is compact.

False. Polynomial Ball, Vector Ball, Discrete Metric

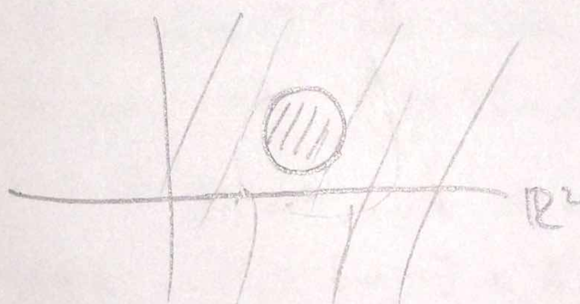
(c) If $A \subset I\!R^2$ is compact then $I\!R^2 - A$ is connected.

False.

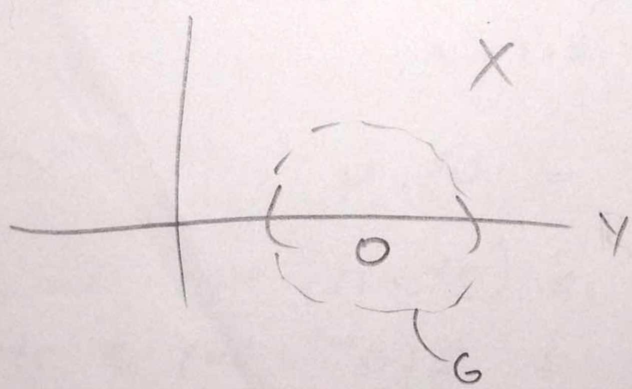
(d) Let Y be a subset of a metric space X with distance d. If O is open in (Y, d), then O is open in (X, d).

False.

c) Suppose A = ball = $\{ x \mid d(0,x) = 1 \}$



d)



3. (10pt)

(a) State the definition of a limit point for a subset E of a metric space X with distance d.

(b) Show that any uncountable subset A of $I\!R$ (with the standard distance) has a limit point.

(a) α is a limit pt. if $\forall \epsilon > 0$

$\exists$ $\gamma \in E$ then $d(\alpha, \gamma) < \epsilon$. ■

(b) $\mathbb{R} = \bigcup^{\infty} [-n, n]$

$\Rightarrow$ $\exists$ $A \cap [-n, n]$ with infinite elements, and bounded. $\Rightarrow$ $\bigcap^{\infty} (A \cap [-n, n])$ where $[-1, 1] \subseteq [-2, 2] \subseteq [\cdots$

$\Rightarrow$ $\exists$ limit pt.

But since $A \cap [-n, n] \subseteq A$, $\Rightarrow$ $A' \neq \phi$.

4. (10pt) Let A and B be subsets of $\mathbb{R}$, and suppose A and B are both bounded above. Show that

(a) The set $A + B$ is bounded above;

(b) $\sup(A + B) = \sup A + \sup B$.

(a) Pf. $\exists \sup A$ and $\exists \sup B$ by the least upper bound property

$\Rightarrow$ $\alpha \in A \leq \sup A \quad \forall \alpha$

$\beta \in B \leq \sup B \quad \forall \beta$

$\Rightarrow$ $\alpha + \beta \leq \sup A + \sup B$

$\Rightarrow$ Because α, β in A, B were chosen arbitrarily, $A+B$ is bounded above. ∎

5. (10pt) Let $\{x_n\}$ be a sequence in $\mathbb{R}^+$.

(a) State the definition of $\limsup x_n$.

(b) Suppose there exists a Cauchy sequence y_n in $\mathbb{R}$ such that $x_n \le y_n$. Using only definition of Cauchy sequence and lim sup, show that $\limsup x_n$ is finite.

(a) $\limsup x_n = \sup \{ \text{subsequential limits of } x_n \}$

(b) $\exists N$ st. $|y_m - y_n| \le \epsilon \quad m, n \ge N$.

$\Rightarrow \quad y_m \le |y_N| + \epsilon$

$\Rightarrow \quad x_n \le |y_N| + \epsilon = M$

$\Rightarrow \quad \limsup x_n = \sup \{ \text{subseq limits of } x_n \} \le M$

$\Rightarrow \quad \limsup x_n \ne \infty$.