Problem 1 (10 pts)

1. Define what it means for a subset S of a vector space V to be linearly dependent.

$S \subseteq V$, subset S spans V, and S generates V.

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2. Let α be the canonical basis of $P_2(\mathbb{R})$ and let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear transformation defined by $T(P) = P'$. Give $[T]_\alpha$

$\alpha = \{1, x, x^2\}$ $\quad T(1) = 1' = 0 = 0\cdot 1 + 0\cdot x + 0\cdot x^2$

$[T]_\alpha^\alpha = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Leftarrow$ $\quad T(x) = x' = 1 = 1\cdot 1 + 0\cdot x + 0\cdot x^2$

$T(x^2) = (x^2)' = 2x = 0\cdot 1 + 2\cdot x + 0\cdot x^2$

✓

3. Define the null space and the range of a linear transformation $T : V \to W$

$W = T(v)$, $N(T)$ means $T(v) = 0$, when $W = 0$

$W = T(v)$ $R(T)$ means all the values that $V \to W$ by linear transformation T.

Problem 2 (10 pts)
Let β be the standard ordered basis for $P_3(\mathbb{R})$ and let γ be an ordered basis on $\mathbb{R}^3$ given by $\gamma = \{(1,2,0),(1,0,1),(0,1,1)\}$. Let T be a linear transformation from $P_3(\mathbb{R})$ to $\mathbb{R}^3$ with matrix representation

$$[T]_\beta^\gamma = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}$$

Compute $T(1+2x+x^2)$. Give your final answer in terms of the standard basis on $\mathbb{R}^3$.

$\beta = \{1, x, x^2\}$ $\quad 1+2x+x^2 = 1\cdot 1 + 2\cdot x + 1\cdot x^2 = (1,2,1)$

$\gamma = \{(1,2,0), (1,0,1), (0,1,1)\}$, $[T]_\beta^\gamma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$

$$[T]_\beta = \begin{bmatrix} 2\cdot(1,2,0) + 0\cdot(1,0,1) + 0(0,1,1) \\ 0\cdot(1,2,0) + 0\cdot(1,0,1) + 2\cdot(0,1,1) \\ 0(1,2,0) + 1\cdot(1,0,1) + 0\cdot(0,1,1) \end{bmatrix}$$

$$= \begin{bmatrix} (2,4,0) \\ (0,2,2) \\ (1,0,1) \end{bmatrix}$$

$$T(1+2x+x^2) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$ 6

$$T(1+2x+x^2) = [2(1,2,0), 1\cdot(1,0,1), 4(0,1,1)]$$

basis: $\beta = \{(2,4,0)_3, (1,0,1), (0,4,4)\}$

Problem 3 (10 pts)
Recall that $P_4(\mathbb{R})$ is the space of polynomials of degree less or equal to 4 in $\mathbb{R}$, Let $V = \{f \in P_4(\mathbb{R}) : f(2) = 0\}$ and $W = \{f \in P_4(\mathbb{R}) : f(1) = 0\}$.

1. Prove that V is a subspace of $P_4(\mathbb{R})$. basis of $P_4(R)$: $\{1, x, x^2, x^3, x^4\}$

$z, y \in V$; $\lambda \in F$;

$\because$ let $f(x) = 0$, $f(2) = 0$. $0 \in V$

$y = a_1 + a_2x + a_3x^2 + a_4x^3 + a_5x^4$. $y(2) = 0 \Rightarrow \lambda y(2) = 0$.

$z = b_1 + b_2x + b_3x^2 + b_4x^3 + b_5x^4$ $z(2) = 0 \Rightarrow \lambda y(2) + z(2) = 0$

$\lambda y + z = \lambda(a_1 + b_1) + \lambda(a_2 + b_2)x + (b_3 + a_3)x^2 + \lambda(a_4 + b_4)x^3 + \lambda(a_5 + b_5)x^4$

$\lambda y(2) + z(2) = 0 \Rightarrow (\lambda y + z)(2) = 0$ $\therefore$ $\lambda y + z \in V$,

2. Prove that $V \cap W$ is a subspace of $P_4(\mathbb{R})$. Hence V is a subspace of $P_4(R)$

proof: $V \cap W = T = \{f \in P_4(R) : f(2) = 0, f(1) = 0\}$.

$y, z \in V \cap W$, $\lambda \in F$.

$y = a_1 + a_2x + a_3x^2 + a_4x^3 + a_5x^4$

$y(1) = y(2) = 0$.

$z = b_1 + b_2x + b_3x^2 + b_4x^3 + b_5x^4$ $z(1) = z(2) = 0$

$(\lambda y + z)(1) = \lambda y(1) + z(1) = 0$. , $(\lambda y + z)(2) = \lambda y(2) + z(2)$

$= \lambda \cdot 0 + 0 = 0$.

$\therefore (\lambda y + z) \in (V \cap W)$ ✓ 10

$\therefore$ $V \cap W$ is a subspace of $P_4(R)$

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Problem 4 (10 pts)
Let V and W be two vector spaces, let β be a basis of V and let T and U be two linear transformations form V to W. Prove that if T and U are equal on β then $T = U$.

proof: T and U are linear transformation from $V \to W$.

V and W are two vector space

DON'T ASSUME FINITE

$\beta = \{v_1, v_2, \ldots, v_n\}$ $W = \{w_1, w_2, \ldots w_n\}$?? W IS A V.S., DEFINITELY NOT FINITE.

I have vectorx, $v \in V$, $\Rightarrow v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$

$T(0) = 0$, $U(0) = 0$

for T and U are equal.

$T(v) = U(v)$

$T(v) = T(a_1 v_1 + a_2 v_2 + \cdots + a_n v_n)$

$= a_1 T(v_1) + a_2 T(v_2) + \cdots + a_n T(v_n)$

$U(v) = U(a_1 v_1 + a_2 v_2 + \cdots + a_n v_n)$

$= a_1 U(v_1) + a_2 U(v_2) + \cdots + a_n U(v_n) = T(v)$

Hence. $T = U$ then on β $T = U$.

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diagonal matrix:

$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

Problem 5 (10 pts)
Let $D_{2\times 2}(\mathbb{R})$ be the space of diagonal matrices in $M_{2\times 2}(\mathbb{R})$. Let $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be the linear transformation given by

$$T(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} c & b \\ a & d \end{pmatrix}$$

Let

$$V = \{v \in M_{2\times 2}(\mathbb{R}) : T(v) \text{ is a diagonal matrix}\}$$

.

T(V) is a diagonal matrix

1. Prove that V is a subspace of $M_{2\times 2}(\mathbb{R})$

proof: $x, y \in V$ $\quad x = \begin{bmatrix} 0 & 0 \\ x_3 & x_4 \end{bmatrix}, y = \begin{bmatrix} 0 & 0 \\ y_3 & y_4 \end{bmatrix}$

$\lambda x + y = \begin{bmatrix} 0 & 0 \\ \lambda x_3 + y_3 & \lambda x_4 + y_4 \end{bmatrix}$ $\quad T(\lambda x + y) = \begin{bmatrix} \lambda x_3 + x_3 & 0 \\ 0 & \lambda x_4 + y_4 \end{bmatrix} \in M_{2\times 2}(\mathbb{R})$

$T(0) = T(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in M_{2\times 2}\mathbb{R}$

Hence, V is subspace of $M_{2\times 2}(\mathbb{R})$

2. Find a basis of V.

$V = \{ v \in M_{2\times 2}(\mathbb{R}) ; T(v) \text{ is a diagonal matrix} \}$

let β is a basis for V

$T\begin{pmatrix} a & d \\ c & b \end{pmatrix} = \begin{pmatrix} c & b \\ a & d \end{pmatrix}$ with $a = b = 0$. YES

$T\begin{pmatrix} a & d \\ c & b \end{pmatrix} = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} = c\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\beta = \{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \} \leftarrow$ basis

NO: WE WANT A BASIS FOR V, NOT T(V)

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