Math 115A | Midterm II | Feb 26, 2020

Name: _______________________________

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Instructions: You have 50 minutes to complete this exam. There are five problems worth a total of 50 points. This exam is closed book and closed notes, and calculators are not allowed. You must justify your answers and show all of your work to receive full credit. Simplify your answers as much as possible. You may lose points for answers that are not simplified. Write your solutions in the space below each question. If your answer continues onto another page, write an easily visible note under the original question. You may use the last two pages of the exam for scratch work.

Question	Points	Score
1	8	8
2	12	12
3	10	10
4	10	10
5	10	10
Total:	50	50

8

1. Write out the following definitions.

(a) (2 points) A **linear map** $T: V \to W$.

Let V, W be vector spaces over a field $\mathbb{F}$.
A function $T: V \to W$ is a linear map if:
(1) $T(x+y) = T(x) + T(y)$ for all $x, y \in V$
(2) $T(cx) = cT(x)$ for all $c \in \mathbb{F}$, $x \in V$

(b) (2 points) The **nullspace and range** of a linear map $T: V \to W$.

Let $T: V \to W$ be a linear map.
The nullspace of T is $N(T) = \{x \in V \mid T(x) = 0_w\}$
The range of T is $R(T) = \{T(x) \in W \mid x \in V\}$

(c) (2 points) An **eigenvector** of a linear map $T: V \to V$ and its corresponding eigenvalue.

Let $T \in \mathcal{L}(V)$ be linear and V be a finite-dimensional vector space. A non-zero vector $x \in V$ is an eigenvector of T if $T(x) = \lambda x$ for some $\lambda \in \mathbb{F}$.

The scalar λ is the corresponding eigenvalue.

(d) (2 points) The **characteristic polynomial** of a linear map $T: V \to V$.

Let $T \in \mathcal{L}(V)$ be linear and V be a finite-dimensional vector space. A polynomial $p(t) = \det(T - tI_V)$ is the characteristic polynomial of T.

12

2. (12 points) Let V and W be finite-dimensional vector spaces with $\dim V = \dim W$, and let $T\colon V \to W$ be a linear map. Prove that the following are equivalent.

(a) T is one-to-one.

(b) T is onto.

(c) $\operatorname{rank} T = \dim V$.

We will prove (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b): Suppose T is one-to-one. Then, $N(T) = \{0_V\}$.
Hence, using the dimension theorem,

$$\dim(V) = \underbrace{\dim(N(T))}_{=0} + \dim(R(T)).$$

Since $\dim(R(T)) = \dim(V) = \dim(W)$ and $R(T)$ is a subspace of W, then $R(T) = W$.
Hence, T is onto. ✓

(b) $\Rightarrow$ (c): Suppose T is onto. Then $R(T) = W$.
Hence, $\operatorname{rank}(T) = \dim(R(T)) = \dim(W) = \dim(V)$. ✓

(c) $\Rightarrow$ (a): Suppose $\operatorname{rank}(T) = \dim(V)$. Then, $\dim(R(T)) = \dim(V) = \dim(W)$.

Hence, using the dimension theorem,

$$\dim(V) = \dim(N(T)) + \underbrace{\dim(R(T))}_{=\dim(V)}.$$

$$\dim(V) = \dim(N(T)) + \dim(V)$$

So, $\dim(N(T)) = 0$. Hence $N(T) = \{0_V\}$ and T is one-to-one. ✓ $\square$

10

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 0 & b-c \\ c-b & 0 \end{pmatrix}$$

3. Let $T\colon M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be the function $T(A) = A - A^t$.

   (a) (4 points) Prove that T is a linear map.

Let $x \in \mathbb{F}$ and $A, B \in M_{2\times 2}(\mathbb{R})$ such that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $b = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$

$$T(A+B) = T\begin{pmatrix} a+a' & b+b' \\ c+c' & d+d' \end{pmatrix} = \begin{pmatrix} a+a' & b+b' \\ c+c' & d+d' \end{pmatrix} - \begin{pmatrix} a+a' & c+c' \\ b+b' & d+d' \end{pmatrix}$$

$$= \begin{pmatrix} 0 & b+b'-c-c' \\ c+c'-b-b' & 0 \end{pmatrix} = \begin{pmatrix} 0 & b-c \\ c-b & 0 \end{pmatrix} + \begin{pmatrix} 0 & b'-c' \\ c'-b' & 0 \end{pmatrix}$$

$$= T(A) + T(B)$$

$$T(xA) = T\begin{pmatrix} xa & xb \\ xc & xd \end{pmatrix} = \begin{pmatrix} xa & xb \\ xc & xd \end{pmatrix} - \begin{pmatrix} xa & xc \\ xb & xd \end{pmatrix} = \begin{pmatrix} 0 & x(b-c) \\ x(c-b) & 0 \end{pmatrix} = x\begin{pmatrix} 0 & b-c \\ c-b & 0 \end{pmatrix}$$

$$= xT(A)$$

Hence T is linear.

   (b) (6 points) Consider the ordered basis

$$\beta = \left\{ \underset{\beta_1}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}, \underset{\beta_2}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \underset{\beta_3}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}, \underset{\beta_4}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} \right\}$$

   for $M_{2\times 2}(\mathbb{R})$ and compute $[T]_\beta$.

$$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

$$T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 0\beta_1 + 1\beta_2 - 1\beta_3 + 0\beta_4$$

$$T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 0\beta_1 - \beta_2 + 1\beta_3 + 0\beta_4$$

$$T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0$$

$$[T]_\beta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad \square$$

4. (10 points) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map $T(x,y,z) = (x,y,-z)$. Let $\gamma = \{(1,0,0),(0,1,1),(1,0,1)\}$. Compute $[T]_\gamma$ using the change of basis theorem.

10

$$[T]_\gamma = Q^{-1}[T]_\beta Q, \text{ where } Q = [I_V]_\gamma^\beta$$

let β be the standard basis: $\{(1,0,0),(0,1,0),(0,0,1)\}$

$$T(1,0,0) = (1,0,0) = 1(1,0,0) + 0(0,1,0) + 0(0,0,1)$$
$$T(0,1,0) = (0,1,0) = 0(1,0,0) + 1(0,1,0) + 0(0,0,1)$$
$$T(0,0,1) = (0,0,-1) = 0(1,0,0) + 0(0,1,0) - 1(0,0,1)$$

$$[T]_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad [I_V]_\gamma^\beta = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$[I_V]_\beta^\gamma = \left([I_V]_\gamma^\beta\right)^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right) = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right)$$

$$[T]_\gamma = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \underline{\begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & -2 & -1 \end{pmatrix}}. \ \square$$

5. (10 points) Let $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear map

$$T(f(x)) = x^2 f''(x) + f'(x) + f(1).$$

10

Find the characteristic polynomial and eigenvalues of T. Find a basis β for $P_2(\mathbb{R})$ consisting of eigenvectors of T.

Let β be a standard basis for $P_2(\mathbb{R}) = \{1, x, x^2\}$

$$T(1) = x^2(0) + 0 + 1 = 1 = 1 + 0x + 0x^2$$

$$T(x) = x^2(0) + 1 + 1 = 2 = 2(1) + 0x + 0x^2$$

$$T(x^2) = x^2(2) + 2x + 1 = 1 + 2x + 2x^2$$

$$[T]_\beta = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

$$[T]_\beta - \lambda I_v = \begin{pmatrix} 1-\lambda & 2 & 1 \\ 0 & -\lambda & 2 \\ 0 & 0 & 2-\lambda \end{pmatrix}$$

characteristic polynomial : $p(t) = \det(T - tI)$

$$p(\lambda) = \det(T - \lambda I) = \det \begin{pmatrix} 1-\lambda & 2 & 1 \\ 0 & -\lambda & 2 \\ 0 & 0 & 2-\lambda \end{pmatrix} \leftarrow \text{upper triangular}$$

$$= (1-\lambda)(-\lambda)(2-\lambda)$$

characteristic polynomial: $p(\lambda) = (1-\lambda)(-\lambda)(2-\lambda)$

You may use this page for scratch work.

eigenvalues of T: $\lambda = 1, 0, 2$.

$\lambda=1$: $[T]_\beta - 1I_V = \begin{pmatrix} 0 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Then, the is a function $a+bx+cx^2$ such that

$0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & c \end{pmatrix}$

$b=0$
$c=0$
a free

$\{1\}$ is an eigenvector of T.

$\lambda=0$: $[T]_\beta - 0I_V = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Then, there is a function $a+bx+cx^2$ such that

$0 = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a+2b \\ c \\ 0 \end{pmatrix}$

$a=-2b$
$c=0$

$\{-2+x\}$ is an eigenvector of T

$\lambda = 2$

$$[T]_\beta - 2I_V = \begin{pmatrix} -1 & 2 & 1 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 3 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Then, there is a function $a + bx + cx^2$ such that

$$0 = \begin{pmatrix} -1 & 3 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -a + 3b \\ b - c \\ 0 \end{pmatrix}$$

$a = 3b$
b free
$c = b$

$\{3 + x + x^2\}$ is an eigenvector of T

$\beta = \{1, -2 + x, 3 + x + x^2\}$ is a basis for $P_2(\mathbb{R})$ consisting of eigenvectors of T.

β diagonalizes T. $\square$