Math 115A Midterm I Jan 29, 2020

Name: ____________________

UID: ____________________

Signature: ____________________

Instructions: You have 50 minutes to complete this exam. There are five problems worth a total of 50 points. This exam is closed book and closed notes, and calculators are not allowed. You must justify your answers and show all of your work to receive full credit. Simplify your answers as much as possible. You may lose points for answers that are not simplified. Write your solutions in the space below each question. If your answer continues onto another page, write an easily visible note under the original question. You may use the last two pages of the exam for scratch work.

Question	Points	Score
1	8	8
2	10	7
3	12	12
4	12	10
5	8	5
Total:	50	42

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1. Write out the following definitions.

(a) (2 points) A **subspace** W of a vector space V over $\mathbb{R}$.

Let V be a vector space over $\mathbb{R}$.
A subset $W \subset V$ is a subspace if W is a vector space with the addition and scalar multiplication from V.

(b) (2 points) A **linearly independent** subset S of a vector space V over $\mathbb{R}$.

Let V be a vector space over $\mathbb{R}$.
A subset $S \subset V$ is linearly independent if S is not linearly dependent. That is, for any distinct vectors $v_1, \ldots, v_k \in S$ and scalars $c_1, \ldots, c_k \in \mathbb{R}$, if $c_1v_1 + \ldots + c_kv_k = 0$, then $c_1 = \ldots = c_k = 0$.

(c) (2 points) A **finite-dimensional vector space** V over $\mathbb{R}$ and the **dimension** of a finite dimensional-vector space V over $\mathbb{R}$.

Let V be a vector space over $\mathbb{R}$.
V is finite-dimensional if V has a finite basis.

The dimension of a finite-dimensional vector space V, denoted $\dim(V)$, is the number of vectors in any basis of V.

(d) (2 points) Given a vector space V over $\mathbb{R}$ and subspaces W_1 and W_2 of V, the **sum** $W_1 + W_2$.

Let V be a vector space over $\mathbb{R}$.
Let W_1 and W_2 be subspaces of V.
The sum of W_1 and W_2, denoted $W_1 + W_2$, is

$\{x + y \in W_1 + W_2 \mid x \in W_1 \text{ and } y \in W_2\}$.

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2. Let $V = P_2(\mathbb{R})$.

(a) (5 points) Is the set $\{1-x, x-x^2, 1-x-x^2\} \subset V$ linearly independent?

To show that this set is linearly dependent, we must show that for distinct scalars $c_1, c_2, c_3 \in \mathbb{R}$, if $c_1(1-x) + c_2(x-x^2) + c_3(1-x-x^2) = 0$ (*), then $c_1 = c_2 = c_3 = 0$.

$(c_1 + c_3) + (c_1 + c_2 - c_3)x + (-c_2 - c_3)x^2 = 0$

$c_1 + c_3 = 0 \longrightarrow c_1 = 0$

$c_1 - c_2 + c_3 = 0 \rightarrow c_2 = 0$

$c_2 + c_3 = 0 \longrightarrow c_3 = 0$

Since $c_1 = c_2 = c_3 = 0$, then (*) forms a nontrivial linear combination of vectors in the set. ✓

Hence, $\{1-x, x-x^2, 1-x-x^2\} \subset V$ is linearly independent.

(b) (5 points) Find subspaces W_1 and W_2 of V such that $\dim W_1 = 1$ and $\dim W_2 = 2$ but $W_1 + W_2 \neq V$.

✓ If $W_1 \subset W_2$, then $W_1 + W_2 = W_2$ because the vectors in W_1 are all contained in W_2.

Then $\dim(W_1 + W_2) = \dim(W_2)$

Let $W_1 = \{1\}$ and let $W_2 = \{1, x\}$.

Then $\dim(W_1) = 1$ and $\dim(W_2) = 2$,

but $W_1 + W_2 = \{2, x\}$, which does not span V.

?

not subspaces

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3. (12 points) Let V be a vector space over $\mathbb{R}$, let $S \subset V$ be a linearly independent subset, and let $v \in V$. Prove that $S \cup \{v\}$ is linearly dependent if and only if $v \in \text{span}(S)$.

Proof

($\Rightarrow$) Assume that $S\cup\{v\}$ is linearly dependent. Then, there are distinct vectors $u_1, \dots, u_k \in S\cup\{v\}$ and scalars $c_1, \dots, c_k \in \mathbb{R}$, not all zero, such that $c_1u_1 + \dots + c_ku_k = 0$ (*). We want to show that $v \in \text{span}(S)$.

There is one vector in $u_1, \dots, u_k$ that is v; otherwise, (*) will produce a nontrivial linear combination of S, which contradicts the fact that S is linearly independent.

Let $u_1 = v$. Then, $c_1v + c_2u_2 + \dots + c_ku_k = 0$ (**).

c_1 also has to be non-zero, or else (**) will produce a nontrivial linear combination of S, which contradicts the fact that S is linearly independent.

Then, $v = -\frac{c_2}{c_1}u_2 - \dots - \frac{c_k}{c_1}u_k$, which is a linear combination of vectors in S. Therefore, $v \in \text{span}(S)$. ✓

($\Leftarrow$) Assume that $v \in \text{span}(S)$. Then v can be written as a linear combination of distinct vectors $u_1, \dots, u_k \in S$ and scalars $c_1, \dots, c_k \in \mathbb{R}$ such that $v = c_1u_1 + \dots + c_ku_k$. $c_1u_1 + \dots + c_ku_k - v = 0$ is ~~the~~ a nontrivial linear combination of vectors in $S\cup\{v\}$.

Hence, $S\cup\{v\}$ is linearly dependent. $\square$

✓

4. Let $V = M_{2\times 2}(\mathbb{R})$ be the vector space of 2×2 matrices with real entries. Recall that if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V$, its **transpose** is $A^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

(a) (6 points) Show that $W = \{A \in V \mid A = A^t\}$ is a subspace of V.

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To show that W is a subspace of V, we must show that W satisfies the three subspace criteria.

Since $W = \{A \in V \mid A = A^T\}$, then A must be a symmetric matrix such that $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$.

① $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = A^T$, $0 \in V$ ✓

② $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$, $A' = \begin{pmatrix} a' & b' \\ b' & d' \end{pmatrix}$, $A + A' = \begin{pmatrix} a+a' & b+b' \\ b+b' & d+d' \end{pmatrix} = A^T + (A')^T \in W$ ✓

③ $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$, $c \in \mathbb{R}$, $cA = \begin{pmatrix} ca & cb \\ cb & cd \end{pmatrix} = cA^T \in W$. ✓

Thus, W is a subspace of V.

(b) (6 points) Find a basis for W and compute $\dim W$.

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A has to be a symmetric matrix $\begin{pmatrix} a & b \\ b & d \end{pmatrix}$ so that $A = A^T$.

Basis: $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$ ✓

$\dim(W) = 3$ ✓ ← 3 vectors in the basis of W.

$a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$

why is your set of vectors linearly independent?

5. Give an example of each of the following (no justification needed):

(a) (2 points) a set of four distinct vectors that spans $\mathbb{R}^3$;

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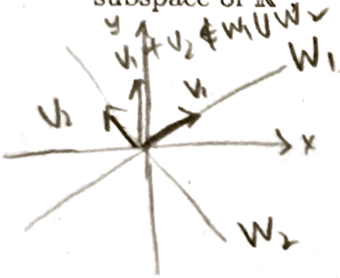
$\{(1,0,0), (0,1,0), (0,0,1), (1,1,1)\}$

(b) (2 points) two subspaces W_1 and W_2 of $\mathbb{R}^2$ such that $W_1 \cup W_2$ is **not** a subspace of $\mathbb{R}^2$;

1



$W_1 = \{(x,y) \mid x,y \in \mathbb{R}^2\}$?

$W_2 = \{(-x,y) \mid x,y \in \mathbb{R}^2\}$?

you may mean $W_1 = \{(x,x) \mid x \in \mathbb{R}\}$, $W_2 = \{(-x,x) \mid x \in \mathbb{R}\}$

(c) (2 points) a subset of $\{1-x, 1+x+x^2, 3+x^2, 2x-2\}$ that is a basis for $P_2(\mathbb{R})$;

2

$\{1-x, 1+x+x^2, 3+x^2\}$

(d) (2 points) a subspace W of $P_2(\mathbb{R})$ such that $P_2(\mathbb{R}) = W \oplus \text{span}(\{1+x\})$.

0

$W = \{ax^2 \mid a \in \mathbb{F}, x \in P_2(\mathbb{R})\}$

$W \oplus \text{span}(\{1+x\}) \neq P_2(\mathbb{R})$

You may use this page for scratch work.

$\{1-x, 1+x+x^2, 3+x^2, 2x-2\}$

$$\begin{pmatrix} 1 & 1 & 3 & -2 \\ -1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 3 & -2 \\ 0 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 1 & 3 & -2 \\ 0 & 1 & 3/2 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 & -2 \\ 0 & 0 & 1/2 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$W \cap \mathrm{span}(\{1+x\}) = \{0\}$

$W + \mathrm{span}(\{1+x\}) = \mathbb{P}_2(\mathbb{R})$

$c_1 x^2 + (c_1 + c_2 x)$