Math 115A | Final Exam | Mar 17, 2020

Name: ______________________________

UID: ______________________________

Signature: ______________________________

Instructions: You have 24 hours to complete this exam, from 12:00 AM to 11:59 PM (Pacific Daylight Time) on Tuesday, March 17, 2020. There are 12 problems worth a total of 120 points. This exam is open book and open notes. You must justify your answers and show all of your work to receive full credit. Simplify your answers as much as possible. You may lose points for answers that are not simplified. Write your solutions in the space indicated on the exam template. If your answer continues onto another page, write an easily visible note under the original question. You may use the last three pages of the exam template for scratch work. If you do not want something you write on the exam template to be graded, you must clearly cross it out. You must scan your completed exam template and upload your solutions to Gradescope by 11:59 PM (Pacific Daylight Time) on Tuesday, March 17, 2020.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	

Question	Points	Score
8	10	
9	10	
10	10	
11	10	
12	10	
Total:	120	

1. Let $V = P_3(\mathbb{R})$.
   (a) (5 points) Prove that $W_1 = \{f(x) \in V \mid f(2) = 0\}$ is a subspace of V.
   (b) (5 points) Is $W_2 = \{f(x) \in V \mid f(2) = 2\}$ a subspace of V? Justify your answer.

2. (10 points) Let $V = \mathcal{F}(\mathbb{R}, \mathbb{R})$. Is $\{e^t, e^{-t}\} \subset V$ linearly independent?

3. (10 points) Find a basis for $V = \{A \in M_{3\times 3}(\mathbb{R}) \mid A^t = -A\}$ and compute $\dim V$. Justify your answer.

4. Let $T\colon P_3(\mathbb{R}) \to P_2(\mathbb{R})$ be the function
$$T(f(x)) = f''(x) - 2f'(x) + f(1).$$
   (a) (3 points) Prove that T is linear.
   (b) (7 points) Let $\beta = \{1, x, x^2, x^3\}$ and $\gamma = \{1 + x, 1 - x, x^2\}$. Compute $[T]_\beta^\gamma$.

5. (10 points) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map, let $\beta = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ be the standard ordered basis for $\mathbb{R}^3$, let $\gamma = \{(-1, 0, 1), (1, 1, 1), (1, -2, 1)\}$ be another ordered basis, and suppose that
$$[T]_\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$
   Use the change of basis theorem to compute $[T]_\beta$.

6. Let V be a finite-dimensional inner product space and let $T, S \in \mathcal{L}(V)$. Let v be an eigenvector of T with eigenvalue λ.
   (a) (5 points) Prove that if $ST = TS$, then $S(v)$ is also an eigenvector of T.
   (b) (5 points) Prove that if λ has algebraic multiplicity 1 as an eigenvector of T, then v is an eigenvector of S.

7. (10 points) Determine whether the matrix
$$A = \begin{pmatrix} 1 & -6 & 4 \\ -2 & -4 & 5 \\ -2 & -6 & 7 \end{pmatrix}$$
   is diagonalizable. If it is, find all of its eigenvalues and a basis for $\mathbb{R}^3$ consisting of eigenvectors of A. If it's not, explain why.

8. (10 points) Let V be a complex inner product space and let $x, y \in V$. Prove that $\langle x, y\rangle = 0$ if and only if $\|x\| \le \|x + cy\|$ for all $c \in \mathbb{C}$.

9. (10 points) Let $V = P_2(\mathbb{R})$ and let $\beta = \{1, x, x^2\}$ be the standard basis for V. Consider the inner product on V defined by

$$\langle f, g \rangle = f(1)g(1) + f(2)g(2) + f(3)g(3). \qquad (*)$$

(You may assume without justification that $(*)$ is an inner product on V.) Use the Gram-Schmidt process on β to produce an orthogonal basis for V.

10. (10 points) Let $V = P_2(\mathbb{R})$ and consider the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)\,dt$$

on V. Find a vector that spans the orthogonal complement of $W = \operatorname{span}\{1+x, x+x^2\}$ in V. Justify your answer.

11. Let V be a finite-dimensional inner product space and let $T \in \mathcal{L}(V)$.
    (a) (4 points) Prove that $N(T^*T) = N(T)$.
    (b) (2 points) Prove that $\operatorname{rank}(T^*T) = \operatorname{rank}(T)$.
    (c) (4 points) Prove that $\operatorname{rank}(T) = \operatorname{rank}(T^*)$.

12. Let V be a finite-dimensional complex inner product space and let $T \in \mathcal{L}(V)$. Recall that the zero operator $T_0 \in \mathcal{L}(V)$ is defined by $T_0(v) = 0_V$ for all $v \in V$.
    (a) (5 points) Suppose that there is some integer $k \geq 1$ such that $T^k = T_0$. Prove that 0 is the only eigenvalue of T.
    (b) (5 points) Now assume that $T^k = T_0$ for some integer $k \geq 1$ and that T is normal. Prove that $T = T_0$.