1	5.
2	3.5
3	5
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T	18.

MATH 115A Midterm I, Spring 2016

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Justify All Your Answers

Problem 1. (5)

Let T be a linear transformation from $\mathbf{R}^4$ to itself defined by $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 1 & 1 & 2 & 3 \\ 3 & 3 & 6 & 9 \\ 4 & 4 & 8 & 10 \\ 2 & 2 & 4 & 6 \end{bmatrix} \end{array}$$

$V_2 = V_1$

$V_3 = 2V_1$

and $\mathbf{x}$ is a column vector in $\mathbf{R}^4$.

(i) Find a basis of the null space of T.

(ii) Find a basis of the range of T.

(iii) What is nullity and rank of T?

$$\begin{pmatrix} 1 & 1 & 2 & 3 \\ 3 & 3 & 6 & 9 \\ 4 & 4 & 8 & 10 \\ 2 & 2 & 4 & 6 \end{pmatrix} \begin{matrix} \\ -3R_1 \\ -4R_1 \\ -2R_1 \end{matrix} \longrightarrow \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} +\frac{3}{2}R_3 \\ \\ \times -\frac{1}{2} \\ \end{matrix} \longrightarrow \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0 \Rightarrow \begin{matrix} x_1 + x_2 + 2x_3 = 0 \\ x_4 = 0 \end{matrix}$$

Let ~~x_3~~ $x_2 = a$ $x_3 = b$

$x_1 = -a - 2b$

~~$(-a-2b, a, b, 0)$~~ $= \begin{pmatrix} -a-2b \\ a \\ b \\ 0 \end{pmatrix} = a\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$\Rightarrow$ basis for $N(T)$: $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

Since V_2, V_3 are linear combinations of V_1, they are redundant

So basis for $R(T) = \{V_1, V_4\} = \left\{ \begin{pmatrix} 1 \\ 3 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 9 \\ 10 \\ 6 \end{pmatrix} \right\}$

iii) $\text{rank}(T) = \dim R(T) = 2$

$\text{null}(T) = \dim N(T) = 2$

$W_1 + W_2 = W_1$

Problem 2. (5)

Let V be a vector space over a field F. Suppose that W_1 and W_2 are two subspaces, neither of them is contained in the other. Prove or disprove the following statements:

(a) $W_1 \cap W_2$ is a subspace;

(b) $W_1 \cup W_2$ is a subspace;

(c) $W_1 + W_2$ is a subspace, where $W_1 + W_2$ is defined to be the collection of elements of the form $z = x + y$ with $x \in W_1$ and $y \in W_2$.

(d) If $W_1 \cap W_2 = \{0\}$, any element z in $W_1 + W_2$ can be uniquely expressed as $z = x + y$ with $x \in W_1$ and $y \in W_2$.

a) Since there is no containment, $W_1 \cap W_2 = \{\emptyset\}$
However $\{\emptyset\}$ cannot be a subspace, since it contains no zero vector, so $W_1 \cap W_2$ is ~~not~~ a subspace.
Addition and Scalar multiplication won't be closed either (no elements). −1

b) No, $W_1 \cup W_2$ is not a subspace, since it will not be closed under addition. Take W_1 to be the x-axis and W_2 to be the y-axis. If $v_1 \in W_1$ and $v_2 \in W_2$ are added, where $v_1, v_2 \neq 0$, then $v_1 + v_2$ will be something not in either axes

$x_1 = y_2 = y_1 = y_2 = 0$
If ~~$z = 0 \in W_1$~~ then $z_1 = z_2 = 0$
$\Rightarrow 0 \in W_1 + W_2$

c) Let $z_1, z_2 \in W_1 + W_2$ s.t. $z_1 = x_1 + y_1$, $z_2 = x_2 + y_2$
Let $c \in F$
$\Rightarrow x_1 \in W_1$, $y_1 \in W_2$, $x_2 \in W_1$, $y_2 \in W_2$

$cz_1 + z_2 = c(x_1 + y_1) + x_2 + y_2 = (cx_1 + x_2) + (cy_1 + y_2)$

$cx_1 + x_2 \in W_1$ and $cy_1 + y_2 \in W_2$

$\Rightarrow (cx_1 + x_2) + (cy_1 + y_2) \in W_1 + W_2$

$\therefore$ $W_1 + W_2$ closed under addition & scalar multiplication. Clearly zero vector is contained, so $W_1 + W_2$ is a subspace

d) This is true, since $W_1 + W_2 = W_1 \oplus W_2 - (W_1 \cap W_2)$
Since $W_1 \cap W_2 = \{0\}$ this becomes $W_1 \oplus W_2$
By definition of direct sum, $z = x + y$ with $x \in W_1$ and $y \in W_2$ (unique x & y)

X −0.5

Problem 3. (5)

Given two matrices $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, define two linear transformations T_A and T_B from $M_{22}(\mathbf{R})$ to itself by the formula $T_A(X) = AX$ and $T_B(X) = BX$ for $X \in M_{22}(\mathbf{R})$. Here $M_{22}(\mathbf{R})$ is the vector space consisting of all 2 by 2 matrices with real entries. Let $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ be the standard basis for $M_{22}(\mathbf{R})$.

(i) Find the matrices $[T_A]_\beta$ and $[T_B]_\beta$.

(ii) Find the matrices $[T_A \circ T_B]_\beta$ and $[T_B \circ T_A]_\beta$.

$\# \; E_{ii}$

(iii) Find the matrix $[(T_A \circ T_B - T_B \circ T_A)^n]_\beta$. Here $(T_A \circ T_B - T_B \circ T_A)^n = (T_A \circ T_B - T_B \circ T_A) \circ (T_A \circ T_B - T_B \circ T_A) \circ \cdots \circ (T_A \circ T_B - T_B \circ T_A)$ is the n-fold composition of $(T_A \circ T_B - T_B \circ T_A)$.

i) $T_A(E_{11}) = \begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix}$ $\quad [T_A(E_{11})]_\beta = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$

$T_A(E_{12}) = \begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix}$ $\quad [T_A(E_{12})]_\beta = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$

$T_A(E_{21}) = \begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix} = \begin{pmatrix}1&0\\0&0\end{pmatrix} = E_{11}$ $\quad [T_A(E_{21})]_\beta = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$

$T_A(E_{22}) = \begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}0&0\\0&1\end{pmatrix} = \begin{pmatrix}0&1\\0&0\end{pmatrix} = E_{12}$ $\quad [T_A(E_{22})]_\beta = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$

$\Rightarrow [T_A]_\beta = \begin{pmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix}$

$T_B(E_{11}) = \begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix} = \begin{pmatrix}0&0\\1&0\end{pmatrix} = E_{21}$ $\quad [T_B(E_{11})]_\beta = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$

$T_B(E_{12}) = \begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix} = \begin{pmatrix}0&0\\0&1\end{pmatrix} = E_{22}$ $\quad [T_B(E_{12})]_\beta = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

$T_B(E_{21}) = 0 \qquad T_B(E_{22}) = 0 \Rightarrow [T_B]_\beta = \begin{pmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\0&1&0&0\end{pmatrix}$

ii) $[T_A \circ T_B]_\beta = [T_A]_\beta [T_B]_\beta = \begin{pmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\0&1&0&0\end{pmatrix} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$

$[T_B \circ T_A]_\beta = [T_B]_\beta [T_A]_\beta = \begin{pmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix} = \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$

iii) $[(T_A \circ T_B) - (T_B \circ T_A)]_\beta = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$

$[(T_A \circ T_B - T_B \circ T_A)^n]_\beta = \left([(T_A \circ T_B - T_B \circ T_A)]_\beta\right)^n = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}^n = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&(-1)^n&0\\0&0&0&(-1)^n\end{pmatrix}$

Since it's a diagonal matrix

$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$

$\Rightarrow a = d = 0$

$b = -c$

Skew symmetric

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & c \\ c & d \end{pmatrix} = ($

Problem 4. (5)

Let $V = M_{22}(\mathbf{R})$ be the vector space of 2 by 2 matrices and $T : V \to V$ be a linear transformation defined by $T(X) = X + X^t$ for $X \in V$. Here X^t is the transpose of the matrix X.

(i) What are the null space $N(T)$ and range $R(T)$ of T?

(ii) Find a basis for $N(T)$ and $R(T)$ respectively. What are the nullity and rank of T?

(iii)What are the interesction $N(T) \cap R(T)$ and the sum $N(T) + R(T)$?

i) $N(T)$: $T(X) = 0 = X + X^t$ $\quad -X = X^t$

$N(T) = \{X \in M_{22}(\mathbb{R}) \mid X^t = -X\}$ (set of skew-symmetric matrices)

Let A be arbitrary matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$T(A) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 2a & b+c \\ b+c & 2d \end{pmatrix} = \begin{pmatrix} x & z \\ z & y \end{pmatrix}$

$x = 2a \quad y = 2d$

$z = b + c$

$\Rightarrow R(T) = \{X \in M_{22}(\mathbb{R}) \mid X = X^t\}$ (symmetric matrices)

ii) $N(T) =$ Skew symmetric matrices $= \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} = b\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\Rightarrow$ basis: $\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$

$R(T)$ = symmetric matrices $= \begin{pmatrix} x & z \\ z & y \end{pmatrix} = x\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + y\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + z\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

basis: $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$

null $(T) = 1$ $\quad$ rank $(T) = 3$

−0.5.

iii) $N(T) \cap R(T) = \{0\}$ since bases do not contain each other ($\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ cannot be written as a combination of elements in basis of $R(T)$ & vice versa)

$N(T) + R(T) = M_{2\times 2}(\mathbb{R})$

$k_1 = b + z \quad k_2 = -b + z$

$\begin{pmatrix} x & z \\ z & y \end{pmatrix} + \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} = \begin{pmatrix} x & b+z \\ -b+z & y \end{pmatrix} = \begin{pmatrix} x & k_1 \\ k_2 & y \end{pmatrix}$ = arbitrary 2×2 matrix