1(a) Prove that homogeneous system of linear equations

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$$a_{11}x_1 + \quad \cdots \quad + a_{1n}x_n = 0$$

n unknowns
m equations

$$a_{m1}x_1 + \cdots \quad + a_{mn}x_n = 0$$

with $n > m$ has a "nontrivial" solution, i.e. there is an $(x_1, \ldots, x_n) \neq (0, 0 \ldots, 0)$ which satisfies the system.

(b) Explain how part (a) shows that if a vector space V is generated by m vectors, then every set of n vectors in V is linearly dependent, $n > m$.

a. We induct on m.

Base case: $m = 1$, so there are at least 2 variables. Let $x_1 \neq 0$ and $x_2 = \frac{a_{11}x_1}{a_{12}}$ if $a_{12} \neq 0$, else let $x_2 \neq 0$ arbitrarily, and $x_1 = 0$.

Inductive step: Suppose we have $m+1$ equations, WLOG we have at least one $a_{m+1,i} \neq 0$ (Else we have m equations and n unknowns and are done by inductive hypothesis). We can then subtract $\frac{a_{j,i}}{a_{m+1,i}}$ times the $m+1$ th equation for all $j \leq m$. Now we have eliminated the ith variable in the first m equations so by the inductive hypothesis there is a nontrivial soltn. $(x_1, \ldots, x_n)$ skipping x_i. Substituting back in, we have $a_{m+1,i}x_i + b = 0$ where b is some constant. This has soltn $\frac{b}{a_{m+1,i}}$ as by assumption $a_{m+1,i} \neq 0$. Since one of the $x_j \neq 0$ w/ $j \neq i$, we have a nontrivial soltn.

(n-1 > m-1)

back →

6. Let $\{v_1, \dots, v_m\}$ be those m vectors. Suppose we have $\{u_1, \dots, u_n\}$ w/ $n > m$. Since the v's generate the space

$\forall i \in \{1, \dots, n\} \; \exists \alpha_{i,j}$ s.t. $\sum_{j=1}^{m} \alpha_{i,j} v_j = u_i$

So

$$\sum_k \lambda_k u_k = 0 \iff \sum_k \lambda_k \left(\sum_{j=1}^{m} \alpha_{k,j} v_j \right) = 0.$$

Exchanging the order of the sum.

we have m equations $\sum_k \lambda_k \alpha_{kj} = 0$

w/ n variables $\lambda_1, \dots, \lambda_n$.

$\therefore$ [illegible] this has nontrivial soln. in the λ's.

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2. Prove: If W_1 and W_2 are finite-dimensional subspaces of a vector space V then $W_1 + W_2$ is finite-dimensional and

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

Let $B = \{u_1, \dots, u_m\}$ be a basis of $W_1 \cap W_2$. We can extend B to a basis of W_1: $\{t_1, \dots, t_\ell, u_1, \dots, u_m\}$, let $A = \{t_1, \dots, t_\ell\}$. We can extend B to be a basis of W_2 containing B: $\{u_1, \dots, u_m, v_1, \dots, v_n\}$. Let $C = \{v_1, \dots, v_n\}$. We claim that $A \cup B \cup C = \{t_1, \dots, t_\ell, u_1, \dots, u_m, v_1, \dots, v_n\}$ is a basis of $W_1 + W_2$.

First, $A \cup B \cup C$ generates $W_1 + W_2$ because its subset $A \cup B$ generates W_1 and $B \cup C$ generates W_2, so $A \cup B \cup C$ generates any sum of elements from the two (take the coefficients on elements of A and C and the sum of coefficients on B. +

$A \cup B \cup C$ is independent. Let $\alpha_1, \dots, \alpha_\ell, \beta_1, \dots, \beta_m, \gamma_1, \dots, \gamma_n \in F$ s.t.

$$\sum_{i=1}^{\ell} \alpha_i t_i + \sum_{i=1}^{m} \beta_i u_i + \sum_{i=1}^{n} \gamma_i v_i = 0$$

$$\sum_i \alpha_i t_i + \sum \beta_i u_i = -\sum_i \gamma_i v_i$$

The RHS $\in W_2$ and LHS $\in W_1$, so LHS, RHS $\in W_1 \cap W_2$.

But if C generates a nonzero element of $W_1 \cap W_2$, then $B \cup C$ is dependent, but $B \cup C$ is a basis of W_2. So RHS = LHS = 0.

By independence of $B \cup C$'s subsets, $\gamma_i = 0$ for $i \leq n$

By independence of $A \cup B$, $\alpha_i = 0$, $\beta_i = 0$ $\forall i$

So $A \cup B \cup C$ is independent.

We have by inclusion-exclusion

$$|A \cup B \cup C| = |A \cup B| + |B \cup C| + |B|$$

which implies the desired statement.

3. Show that a square matrix A is invertible if and only if $\det A \neq 0$ by considering the columns and operations on them.

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0. Exchanging columns flips the sign of the determinant and therefore does not change the nonzeroness of the determinant.

Det(A) is multilinear in the col. of A, so multiplying a col. by a nonzero constant does not change whether the det. is 0.

Again det(A) is multilinear in A's columns, so Adding a column to another is equivalent to adding the det.( ) matrix w/ the adding column copied. Since det is alternating, that is 0. So Adding one column to another does not change the determinant.

These operations are sufficient to perform Gaussian elimination on the matrix. By the above arguments $\det(RREF(A)) \neq 0$ iff $\det(A) \neq 0$. But A is invertible iff $RREF(A) = I$, so. A invertible iff $\det(A) \neq 0$.

✓

4. (a) Define the annihilator W^a of a subspace of V ($\dim V = n < +\infty$) $(W^a \subset V^*)$

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(b) Show that $\dim W^a = n - \dim W$

(c) Combine part(b) with the rank and nullity theorem to show for a ~~symm~~ matrix M (not necessarily square!)

row rank = column rank

2/2

a. $W^a = \{\theta \in V^* : \forall w \in W\ \theta(w) = 0\}$ ✓

b. Let $\{u_1, \dots, u_m\}$ be a basis for W. We can for a basis for V containing

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that $\{u_1, \dots, u_m, u_{m+1}, \dots u_n\}$. Let $B^* = \{\theta_1, \dots, \theta_m, \theta_{m+1}, \dots \theta_n\}$ be the corresponding dual basis (basis of V^*). We claim that $C = \{\theta_{m+1}, \dots, \theta_n\}$ is a basis for W^a. As it is a subset of a basis, it is independent. Let $\theta \in W^a$. We have

$$\theta = \sum_{i=1}^{n} \theta(u_i)\theta_i$$

Since $\forall i \le m$, $u_i \in W$; $\forall i \le m\ \theta(u_i) = 0$

$$\theta = \sum_{i=m+1}^{n} \theta(u_i)\theta_i$$, so C generates W^a.

+ $C \subset W^a$?

$\dim W^a = |C| = n - m = n - \dim W$ ∎

0/3

c.

5. (a) Define the trace of a square matrix M $\left(\begin{matrix}\text{notation} \\ \operatorname{tr} M\end{matrix}\right)$

(b) If A, B are two square $n \times n$ matrices, show $\operatorname{tr}(AB) = \operatorname{tr}(BA)$

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(c) Use part (b) to show that if P is $n \times n$ invertible, then

$$\operatorname{tr}(P^{-1}AP) = \operatorname{tr}(A)$$

for every $n \times n$ matrix A, invertible or not.

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a. $\operatorname{tr} M = \sum_i (M)_{ii}$ ✓

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b. $\operatorname{tr}(AB) = \sum_i (AB)_{ii} = \sum_i \left(\sum_j (A)_{ij} (B)_{ji} \right)$

$\operatorname{tr}(BA) = \sum_i \sum_j (A)_{ji} (B)_{ij}$

exchanging order of sums and reindexing w/ $i' = j$, $j' = i$

$\operatorname{tr}(BA) = \sum_{i'} \sum_{j'} (A)_{i'j'} (B)_{j'i'} = \operatorname{tr}(AB)$. ✓

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c. $\operatorname{tr}((P^{-1}A)P) = \operatorname{tr}(P(P^{-1}A))$ by part b

by associativity

$= \operatorname{tr}((PP^{-1})A) = \operatorname{tr}(A)$. ✓

6. Explain why, if $(a,b,c) \neq (0,0,0)$, $a,b,c \in \mathbb{Q}$ then there are numbers $x,y,z \in \mathbb{Q}$ (rational numbers) such that

$$\left(a + b\sqrt[3]{2} + c\sqrt[3]{4}\right)\left(x + y\sqrt[3]{2} + z\sqrt[3]{4}\right) = 1$$

[You may use any of the general properties of polynomials that we covered in homework if you quote them precisely]