Math 115AH Midterm 1

1. (20 points) Let V be a nonzero vector space over a field F and S a nonempty subset of vectors in V. Fully and accurately defined what it means for each of the following:
   - (a) A vector v in V to be in the **Span** of S.
   - (b) S to be a **linearly dependent** subset in V.
   - (c) S to be a **linearly independent** subset in V.
   - (d) $V = \sum_{i \in I} W_i$ where each W_i, $i \in I$, is a subspace of V.

2. (30 points) Give examples of each of the following (You do **not** need to justify):

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} -a & -c \\ -b & -d \end{pmatrix}$

$a = d = 0$

$b = -c$

   - (a) Let $\mathbb{M}_2(F)$ be the vector space over F of 2×2 matrices. Find a **basis** for the subspace of skew symmetric matrices $\{A \in \mathbb{M}_2(F) \mid A^t = -A\}$ in each of the following two cases:
     - (i) F is the field of two elements.
     - (ii) F is the field of three elements.
   - (b) Let $V = C([0,1])$ be the real vector space of continuous functions on $[0,1]$. Find a spanning set for V and two subspaces W and W_6 of V and a basis for each with W infinite dimensional (i.e., is not finite dimensional) and W_6 finite dimensional of dimension 6
   - (c) A **vector space** V and **subspaces** $\{0\} \neq W_i$ of V with $W_i \neq V$ for $i = 1, 2$ with W_1 infinite dimensional and isomorphic to V and W_2 four dimensional.
   - (d) **Linear transformations** $T : V \to W$ and $S : W \to U$ of nonzero vector spaces such that the composition $S \circ T$ is an isomorphism, but neither S nor T is an isomorphism.
   - (e) A linear operator $T : V \to V$ that is onto but has a non-trivial kernel, i.e., $\ker T \neq 0$.

3. (35 points) Do all of the following:
   - (a) Accurately **state** the **Replacement Theorem** and one other **(named) theorem** that we have proved about vector spaces.
   - (b) Accurately **state** the **Isomorphism Theorem** and one other **(named) theorem** that we have proved about linear transformations.
   - (c) Fully state one **consequence** (i.e., corollary, proposition, or theorem) of one of the theorems that you wrote in (a) and one **consequence** of the theorems you wrote in (b). (You do **not** need to justify.)
   - (d) Prove the Isomorphism Theorem.

4. (15 points) Let V be a finite dimensional vector space over F and $V = W_1 + \cdots + W_n$ with each W_i a nonzero subspace of V, $i = 1, \ldots, n$. Prove the following are equivalent:
   - (a) If $v \in W$ then there exist unique $w_i \in W_i$, $i = 1, \ldots, n$, satisfying $v = w_1 + \cdots + w_n$.
   - (b) $W_i \bigcap (W_1 + \cdots + \widehat{W_i} + \cdots + W_n) = \{0\}$, for $i = 1, \ldots, n$, where $\widehat{W_i}$ means omit W_i.
   - (c) If $\mathcal{B}_i$ is a basis for W_i, $i = 1, \ldots, n$, then $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_n$ is a basis for W.✓

   [If these hold we write $V = W_1 \oplus \cdots \oplus W_n$ and say V is a the **direct sum** of the W_i's.]

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5. Redo the test at home. Turn it in next Friday in class.