Midterm 2

UCLA: Math 115AH, Winter 2019

Instructor: Jens Eberhardt
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- This exam has 3 questions, for a total of 20 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: [redacted]

ID number: [redacted]

Question	Points	Score
1	6	
2	6	
3	8	
Total:	20	

1. Let F be a field, V, W be a vector spaces over F and $T : V \to W$ be a linear maps. Recall that the *dual* of or *pullback* along T is defined by

$$T^* : W^* \to V^*, \alpha \mapsto (V \to F, v \mapsto \alpha(v)).$$

Now let U be another vector space over F and $S : U \to V$ a linear map.

(a) (2 points) Show that $(TS)^* = S^*T^*$.

$$\begin{tikzcd} U \arrow[r, "\alpha TS"] \arrow[d, "S"'] & F \\ V \arrow[r, "T"'] \arrow[ur, "\alpha T"] & W \arrow[u, "\alpha"'] \end{tikzcd}$$

$S^* : V^* \to U^*$ $\quad$ $TS : U \to W$

$S^*T^* : W^* \to U^*$ $\quad$ $(TS)^* = W^* \to U^*$.

$S^*(T^*(\alpha))(v) = S^*(\alpha T)(v) = \cancel{\alpha T}\ \alpha(T(S(v)))$, $v \in U$

$(TS)^*(\alpha)(v) = \alpha(T(S(v)))$ $v \in U$ $\qquad$ $\alpha(T(S(v))) = \alpha(T(S(v)))$

So $S^*T^* = (TS)^*$.

(b) (2 points) Assume that T is invertible. Give an inverse of T^* in terms of T^{-1}. Prove that your inverse is really an inverse.
Hint: What is id_V^* and id_W^*?

$T^{-1} : W \to V$

$T^{*-1} : V^* \to W^*$, $\varphi \mapsto (W \to F, w \mapsto \varphi T^{-1}(w))$

$$\begin{tikzcd} & F \\ W \arrow[r, "T^{-1}"'] \arrow[ur, "\varphi T^{-1}"] & V \arrow[u, "\varphi"'] \end{tikzcd}$$

or if $\alpha T \in V^*$

$T^{*-1} : \alpha T \mapsto \alpha(T(T^{-1})))$

$\alpha T \mapsto \alpha$.

$T^*(T^{*-1}(\alpha T))$, $\alpha T \in V^*$

$T^*(T^{*-1}(\alpha T)) = T^*(\alpha(T(T^{-1}))) = T^*(\alpha) = \alpha T$

$T^{*-1}(T^*(\alpha)) = T^{*-1}(\alpha T) = \alpha(T(T^{-1})) = \alpha$

Thus $T^{*-1} : V^* \to W^*$

(c) (2 points) Show that $\mathrm{im}(T)^*$ is isomorphic to $(V/\ker(T))^*$.
Hint: Dimension arguments do not work here. Use (b).

$\mathrm{im}(T)^* : \mathrm{Hom}(\mathrm{im}(T), F)$.

$\mathrm{im}(T) = \{w \in W \mid T(v) = w, \forall v \in V\}$

$V/\ker(T) \cong \mathrm{Hom}(V/\ker(T), F)$.

$L: \mathrm{Hom}(\mathrm{im}(T), F) \longrightarrow \mathrm{Hom}(V/\ker(T), F)$.

Let $\tilde{L} : \cancel{\mathrm{im}(T)}\ V/\ker(T) \to \cancel{V/\ker(T)}\ \mathrm{im}(T)$

by isomorphism th.

$$\begin{tikzcd} V/\ker(T) \arrow[r, "\tilde{L}"] \arrow[dr, "\alpha\tilde{L}"'] & \mathrm{im}(T) \arrow[d, "\alpha"] \\ & F \end{tikzcd}$$

$\mathrm{im}(T)$ is isomorphic to $V/\ker(T)$.

Let $\alpha \in \mathrm{im}(T)^*$.

$L(\alpha) = \alpha\tilde{L} \in (V/\ker(T))^* \rightsquigarrow$ since $\tilde{L} : V/\ker(T) \cong \mathrm{im}\, T$ is isomorphic

Using isomorphism theorem; $\mathrm{im}(T) \cong V/\ker(T)$

Therefore, $(\mathrm{im}(T))^* \cong (V/\ker(T))^*$

2. Let F be a field and V be a vector space over F. Recall that for an endomorphism $T \in \mathrm{End}(V)$ and $\lambda \in F$ we denote

$$E_\lambda(T) = \{v \in V \,|\, T(v) = \lambda v\} = \ker(T - \lambda\,\mathrm{id}_V) \subseteq V.$$

(a) (2 points) Show that for $\lambda \in F$ the space $E_\lambda(T)$ is invariant under T.

by definition $E_\lambda(T) = \{v \in V \mid T(v) = \lambda v\} \subseteq V$.

Let $T(v) = \lambda v \in \text{span}\{v \in V \mid T(v) = \lambda v\}$

So $T(T(v)) = \lambda^2 v \in \text{span}\{v \in V \mid T(v) = \lambda^2 v\}$
since $\lambda \in F$.

Thus $E_\lambda(T)$ is invariant under T

Since if $v \in E_\lambda(T)$,
$T(v) \in E_\lambda(T)$

(b) (2 points) Show that

$$E_\lambda(T) \subseteq E_{\lambda^2}(T^2).$$

if $v \in E_\lambda(T)$, $v \in E_{\lambda^2}(T^2)$.

if $v \in E_\lambda(T)$

$T(v) = \lambda v$.

$T(\lambda v) = \lambda T(v)$ by linearity.

So $T(T(v)) = T(\lambda(v)) = \lambda T(v) = \lambda^2 v$.

$T(T(v)) = T^2(v)$ so $T^2(v) = \lambda^2 v$, $v \in E_\lambda$

So if $v \in E_\lambda(T)$, $v \in E_{\lambda^2}(T^2)$.

So $E_\lambda(T) \subseteq E_{\lambda^2}(T^2)$.

(c) (2 points) Let $\lambda_1, \lambda_2 \in F$. Assume that $E_{\lambda_1}(T) \cap E_{\lambda_2}(T) \neq \{0\}$. Show that $\lambda_1 = \lambda_2$.

Let $v_1 \in E_{\lambda_1}(T)$ s.t. $T(v_1) = \lambda_1 v_1$

Also let $v_2 \in E_{\lambda_2}(T)$ s.t. $T(v_2) = \lambda_2 v_2$.

Assuming that $E_{\lambda_1}(T) \cap E_{\lambda_2}(T) \neq 0$

~~Assume~~ Assume $v_1 = v_2 \in E_{\lambda_1}(T) \cap E_{\lambda_2}(T)$.

So $T(v_1) = \lambda_1 v_1$

$T(v_2) = \lambda_1 v_2$ since $v_1 = v_2$. by construction.

but $T(v_2) = \lambda_2 v_2$.

and $T(v_2) = T(v_2)$ ~~since~~ by identity axiom.

So $\lambda_1 v_2 = \lambda_2 v_2 \Rightarrow \lambda_1 v_2 - \lambda_2 v_2 = 0$

So $v_2(\lambda_1 - \lambda_2) = 0$, and since by definition $v_2 \neq 0$

$\lambda_1 - \lambda_2 = 0 \Rightarrow \lambda_1 = \lambda_2$

3. Let V be an inner product space over $F = \mathbb{R}$ where the inner product is denoted by $\langle x, y\rangle$ for $x, y \in V$. Consider the map

$$\ell : V \to V^*, x \mapsto (V \to F, y \mapsto \langle x, y\rangle).$$

(a) (2 points) Show that ℓ is linear.

Let $x, z \in V$, $\lambda \in F$.

prove: $\ell(\lambda x + z) = \lambda \ell(x) + \ell(z)$.

RHS: $\ell(\lambda x + z) = \langle \lambda x + z, y\rangle = \lambda \langle x, y\rangle + \langle z, y\rangle$ since inner product is sesqui linear.

LHS: $\lambda \ell(x) + \ell(z) = \lambda \langle x, y\rangle + \langle z, y\rangle = $ RHS.

Since RHS = LHS $\ell(\lambda x + z) = \lambda \ell(x) + \ell(z)$.

So ℓ is linear.

(b) (2 points) Show that ℓ is injective.

To prove injectivity wts if $\ell(x) = \ell(z)$

$z = x$, $x, z \in V$

RHS: $\ell(x) = \langle x, y\rangle$

LHS: $\ell(z) = \langle z, y\rangle$.

Assume $\ell(z) = \ell(x)$, so $\langle x, y\rangle = \langle z, y\rangle$

~~wts.~~ this implies, $\langle x, y\rangle - \langle z, y\rangle = 0$.

$\Rightarrow \langle x - z, y\rangle = 0$ since inner product sesqui linear

Lemma: $\langle 0_v, y\rangle \Rightarrow \langle 0_F 0_v, y\rangle \Rightarrow 0_F \langle 0_v, y\rangle$
$= 0$.

So $\langle 0_v, y\rangle = 0$. So

$\langle x - z, y\rangle = \langle 0_v, y\rangle = 0$ So $x - z = 0$

(c) (4 points) Now let $T : V \to V^*$ is a linear map. Define

$$\langle -, - \rangle_T : V \times V \to F, (x, y) \mapsto T(x)(y).$$

So in other words, $\langle x, y \rangle_T = T(x)(y)$. Assume that T is injective, $T(x)(y) = T(y)(x)$, and that $T(x)(x) \geq 0$ for all $x, y \in V$.
Show that $\langle -, - \rangle_T$ is an inner product.

Axioms of inner product.

1) $\langle \lambda x + y, z \rangle = \lambda \langle x, z \rangle + \langle y, z \rangle$, $x, y, z \in V$.

2) $\langle x, x \rangle \geq 0$, equality when $x = 0$.

3) $\langle x, y \rangle = \overline{\langle y, x \rangle} = \langle y, x \rangle$ if $F = \mathbb{R}$.

2) → Since by definition $T(x)(x) \geq 0$, the second axiom holds, and $T(x)(x) > 0$ $\forall x \neq 0$.

3) → Since $T(x)(y) = T(y)(x)$, the third axiom holds since $F = \mathbb{R}$.

1) → WTS Linear in first "space" so wts $T(\lambda x + y)(z) = \lambda T(x)(z) + T(y)(z)$ $\lambda \in F$, $x, y, z \in V$.

Since T by definition is a linear map

$T(\lambda x + y)(z) = \lambda T(x)(z) + T(y)(z)$

so $\langle -, - \rangle_T$ is sesquilinear.

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