

# Midterm 2

## UCLA: Math 115AH, Winter 2019

*Instructor: Jens Eberhardt*  
*Date: 1 March 2019*

- This exam has 3 questions, for a total of 20 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

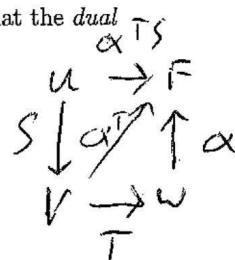
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Question	Points	Score
1	6	
2	6	
3	8	
Total:	20	

1. Let  $F$  be a field,  $V, W$  be vector spaces over  $F$  and  $T : V \rightarrow W$  be a linear map. Recall that the *dual* of or *pullback* along  $T$  is defined by

$$T^* : W^* \rightarrow V^*, \alpha \mapsto (V \rightarrow F, v \mapsto \alpha(v)).$$



Now let  $U$  be another vector space over  $F$  and  $S : U \rightarrow V$  a linear map.

- (a) (2 points) Show that  $(TS)^* = S^*T^*$ .

$$S^* : V^* \rightarrow U^*$$

$$TS : U \rightarrow W$$

$$S^*T^* : W^* \rightarrow U^*$$

$$(TS)^* : W^* \rightarrow U^*$$

$$S^*(T^*(\alpha))_{(u)} = S^*(\alpha T)_{(u)} = \alpha(T(S(u))), \quad \forall u \in U$$

$$(TS)^*(\alpha)_{(u)} = \alpha(T(S(u))) \quad \forall u \in U \quad \alpha(T(S(u))) = \alpha(T(S(u)))$$

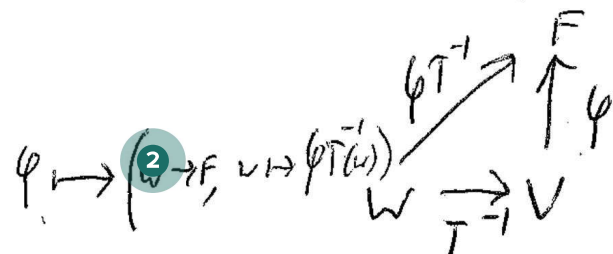
So  $S^*T^* = (TS)^*$

- (b) (2 points) Assume that  $T$  is invertible. Give an inverse of  $T^*$  in terms of  $T^{-1}$ . Prove that your inverse is really an inverse.

Hint: What is  $id_V^*$  and  $id_W^*$ ?

$$T^{-1} : W \rightarrow V$$

$$T^{*-1} : V^* \rightarrow W^*, \quad \varphi \mapsto \varphi \circ T^{-1}$$



or if  $\alpha T \in V^*$

$$T^{*-1}(\alpha T) \mapsto \alpha(T(T^{-1}(v)))$$

$$\alpha T \mapsto \alpha$$

$$T^*(T^{*-1}(\alpha T)) = \alpha T \in V^*$$

$$T^*(T^{*-1}(\alpha)) = T^*(\alpha(T(T^{-1}(v)))) = \alpha(T(v)) = \alpha T$$

$$T^{*-1}(T^*(\alpha)) = T^{*-1}(\alpha T) = \alpha(T(T^{-1}(v))) = \alpha$$

Thus  $T^{*-1} : V^* \rightarrow W^*$

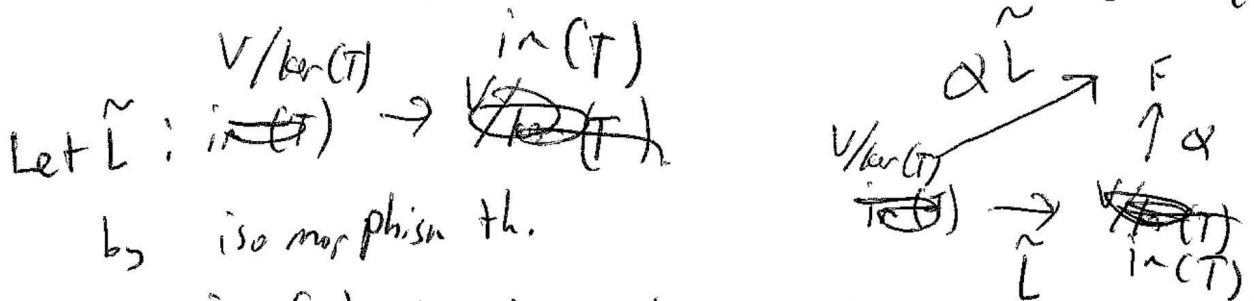
(c) (2 points) Show that  $\text{im}(T)^*$  is isomorphic to  $(V/\ker(T))^*$ .  
 Hint: Dimension arguments do not work here. Use (b).

$$\text{im}(T)^* : \text{Hom}(\text{im}(T), F)$$

$$\text{im}(T) = \{w \in W \mid T(v) = w, \forall v \in V\}$$

$$V/\ker(T) \cong \text{Hom}(V/\ker(T), F)$$

$$L : \text{Hom}(\text{im}(T), F) \longrightarrow \text{Hom}(V/\ker(T), F)$$



Let  $\alpha \in \text{im}(T)^*$ .

$$L(\alpha) = \alpha \tilde{L} \in (V/\ker(T))^* \rightarrow \text{since } \tilde{L} : V/\ker(T) \xrightarrow{\cong} \text{im}(T) \text{ is isomorphic}$$

Using isomorphism theorem;  $\text{im}(T) \cong V/\ker(T)$   
 Therefore,  $(\text{im}(T))^* \cong (V/\ker(T))^*$

2. Let  $F$  be a field and  $V$  be a vector space over  $F$ . Recall that for an endomorphism  $T \in \text{End}(V)$  and  $\lambda \in F$  we denote

$$E_\lambda(T) = \{v \in V \mid T(v) = \lambda v\} = \ker(T - \lambda \text{id}_V) \subseteq V.$$

(a) (2 points) Show that for  $\lambda \in F$  the space  $E_\lambda(T)$  is invariant under  $T$ .

by definition  $E_\lambda(T) = \{v \in V \mid T(v) = \lambda v\} \subseteq V$ .

Let ~~some~~  $T(v) = \lambda v \in \text{span}\{v \in V \mid T(v) = \lambda v\}$

so  $T(T(v)) = \lambda^2 v \in \text{span}\{v \in V \mid T(v) = \lambda v\}$   
since  $\lambda \in F$ .

Thus  $E_\lambda(T)$  is invariant under  $T$

Since if  $v \in E_\lambda(T)$ ,  
 $T(v) \in E_\lambda(T)$

(b) (2 points) Show that

$$E_\lambda(T) \subseteq E_{\lambda^2}(T^2).$$

if  $v \in E_\lambda(T)$ ,  $v \in E_{\lambda^2}(T^2)$ .

if  $v \in E_\lambda(T)$

$$T(v) = \lambda v.$$

$T(\lambda v) = \lambda T(v)$  by linearity.

$$\text{so } T(T(v)) = T(\lambda v) = \lambda T(v) = \lambda^2 v.$$

$$T(T(v)) = T^2(v) \text{ so } T^2(v) = \lambda^2 v, v \in E_\lambda(T)$$

so if  $v \in E_\lambda(T)$ ,  $v \in E_{\lambda^2}(T^2)$ .

so  $E_\lambda(T) \subseteq E_{\lambda^2}(T^2)$ .

(c) (2 points) Let  $\lambda_1, \lambda_2 \in F$ . Assume that  $E_{\lambda_1}(T) \cap E_{\lambda_2}(T) \neq \{0\}$ . Show that  $\lambda_1 = \lambda_2$ .

$$\text{Let } v_1 \in E_{\lambda_1}(T) \text{ s.t. } T(v_1) = \lambda_1 v_1$$

$$\text{Also let } v_2 \in E_{\lambda_2}(T) \text{ s.t. } T(v_2) = \lambda_2 v_2.$$

Assuming that  $E_{\lambda_1}(T) \cap E_{\lambda_2}(T) \neq \{0\}$

~~Assume~~ Assume  $v_1 = v_2 \in E_{\lambda_1}(T) \cap E_{\lambda_2}(T)$ .

$$\text{So } T(v_1) = \lambda_1 v_1 \quad \circ$$

$$T(v_2) = \lambda_1 v_2 \text{ since } v_1 = v_2 \text{ by construction.}$$

$$\text{but } T(v_2) = \lambda_2 v_2.$$

$$\text{and } T(v_2) = T(v_1) \text{ ~~same~~ by identity axiom.}$$

$$\text{So } \lambda_1 v_2 = \lambda_2 v_2 \Rightarrow \lambda_1 v_2 - \lambda_2 v_2 = 0$$

$$\text{So } v_2(\lambda_1 - \lambda_2) = 0, \text{ and since by definition } v_2 \neq 0,$$

$$\lambda_1 - \lambda_2 = 0 \Rightarrow \lambda_1 = \lambda_2$$

3. Let  $V$  be an inner product space over  $F = \mathbb{R}$  where the inner product is denoted by  $\langle x, y \rangle$  for  $x, y \in V$ . Consider the map

$$\ell: V \rightarrow V^*, x \mapsto (V \rightarrow F, y \mapsto \langle x, y \rangle).$$

- (a) (2 points) Show that  $\ell$  is linear.

$$\text{Let } x, z \in V, \lambda \in F.$$

$$\text{prove: } \ell(\lambda x + z) = \lambda \ell(x) + \ell(z).$$

$$\text{RHS: } \ell(\lambda x + z) = \langle \lambda x + z, y \rangle = \lambda \langle x, y \rangle + \langle z, y \rangle$$

since inner product is sesquilinear.

$$\text{LHS: } \lambda \ell(x) + \ell(z) = \lambda \langle x, y \rangle + \langle z, y \rangle = \text{RHS.}$$

$$\text{Since RHS} = \text{LHS} \quad \ell(\lambda x + z) = \lambda \ell(x) + \ell(z).$$

So  $\ell$  is linear.

- (b) (2 points) Show that  $\ell$  is injective.

To prove injectivity wts if  $\ell(x) = \ell(z)$

$$z = x, \quad x, z \in V$$

$$\text{RHS: } \ell(x) = \langle x, y \rangle$$

$$\text{LHS: } \ell(z) = \langle z, y \rangle.$$

$$\text{Assume } \ell(z) = \ell(x), \text{ so } \langle x, y \rangle = \langle z, y \rangle$$

$$\text{this implies } \langle x, y \rangle - \langle z, y \rangle = 0.$$

$$\Rightarrow \langle x - z, y \rangle = 0 \quad \text{since inner product is sesquilinear.}$$

$$\text{Lemma: } \langle 0_v, y \rangle = \langle 0_F 0_v, y \rangle = 0_F \langle 0_v, y \rangle = 0.$$

$$\text{So } \langle 0_v, y \rangle = 0. \text{ So}$$

$$\langle x - z, y \rangle = \langle 0_v, y \rangle = 0 \quad \text{So } x - z = 0$$

(c) (4 points) Now let  $T: V \rightarrow V^*$  is a linear map. Define

$$\langle -, - \rangle_T: V \times V \rightarrow F, (x, y) \mapsto T(x)(y).$$

So in other words,  $\langle x, y \rangle_T = T(x)(y)$ . Assume that  $T$  is injective,  $T(x)(y) = T(y)(x)$ , and that  $T(x)(x) \geq 0$  for all  $x, y \in V$ .

Show that  $\langle -, - \rangle_T$  is an inner product.

Axioms of inner product.

$$1) \langle \lambda x + y, z \rangle = \lambda \langle x, z \rangle + \langle y, z \rangle \\ x, y, z \in V.$$

$$2) \langle x, x \rangle \geq 0, \text{ equality when } x=0.$$

$$3) \langle x, y \rangle = \overline{\langle y, x \rangle} = \langle y, x \rangle \text{ if } F = \mathbb{R}.$$

2)  $\rightarrow$  Since by definition  $T(x)(x) \geq 0$ , the second axiom holds, and  $T(x)(x) > 0 \forall x \neq 0$ .

3)  $\rightarrow$  Since  $T(x)(y) = T(y)(x)$ , the third axiom holds since  $F = \mathbb{R}$ .

1)  $\rightarrow$  w TS Linear, so  $\forall$ ts  $T(\lambda x + y)(z) = \lambda T(x)(z) + T(y)(z)$   
in first "space"  $\lambda \in F, x, y, z \in V$   
 Since  $T$  by definition is a linear map  
 $T(\lambda x + y)(z) = \lambda T(x)(z) + T(y)(z)$   
 so  $\langle -, - \rangle_T$  is sesquilinear.

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