Midterm 2

UCLA: Math 115AH, Fall 2018

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- This exam has 3 questions, for a total of 18 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ______________________________

ID number: ______________________________

Question	Points	Score
1	4	~~0~~ 2
2	6	4
3	8	6
Total:	18	~~10~~ 12

Good Luck!

1. Let F be a field, V be a vector space over F and $W \subset V$ a subspace [redacted] of V. Let

$$\pi : V^* \to W^*, \lambda \mapsto (W \to F, w \mapsto \lambda(w)).$$

denote the restriction map. This is a linear map. Now consider the map

$$T : \ker(\pi) \to (V/W)^*, \lambda \mapsto (V/W \to F, v + W \mapsto \lambda(v)).$$

(a) (2 points) Show that T is well defined. Namely, show that for all $\lambda \in \ker(\pi)$ and $v, v' \in V$ with $v + W = v' + W$ we have

$$\lambda(v) = \lambda(v').$$

(b) (2 points) Give an explicit formula for the inverse of T.

a) let $v, v' \in V$ with $v+W = v'+W$, let $\lambda \in \ker(\pi)$, then

$T(\lambda) = \lambda' : V/W \to F,\ v+W \mapsto \lambda(v)$

Then $\lambda'(v+W) = \lambda(v)$, and $\lambda'(v'+W) = \lambda(v')$

but $\lambda'(v+W) = \lambda'(v'+W)$ that's exactly why we want to prove that $\lambda(v) = \lambda(v')$ so you can't use it, you don't know it.

$\Rightarrow \lambda(v) = \lambda(v')$

b) $T^{-1} : (V/W)^* \to \ker(\pi),\ \lambda' \mapsto (V \to F,\ v \mapsto \lambda'(v+W))$ ✓

2. Let F be a field and V be a vector space over F. Recall that for an endomorphism $T \in \mathrm{End}(V)$ and $\lambda \in F$ we denote

$$E_\lambda(T) = \ker(T - \lambda\,\mathrm{id}_V) \subseteq V.$$

Now let $S, T \in \mathrm{End}(V)$ be endomorphisms of V such that $TS = ST$. So S and T *commute*.

(a) (2 points) Show that for $\lambda \in F$ the space $E_\lambda(T)$ is invariant under S.

(b) (2 points) Let $\lambda \in F$ such that $\dim(E_\lambda(T)) = 1$. Show that there is $\lambda' \in F$ such that

$$E_\lambda(T) \subseteq E_{\lambda'}(S).$$

(c) (2 points) Now assume that T is diagonalizable and all eigenspaces of T have dimension one. Show that also S is diagonalizable.

a) Let $\lambda \in F$, $E_\lambda(T)$ its associated eigenspace, let $v \in V$ s.t.

$S(v) \in E_\lambda(T)$, then $TS(v) = \lambda \cdot S(v)$

Now let $v' \in E_\lambda(T)$, then $ST(v') = S(\lambda v') = \lambda S(v')$

By def. $ST = TS$, thus $\lambda S(v') = \lambda S(v)$?

$\Rightarrow S(v') = S(v)$ No?

As $S(v) \in E_\lambda(T) \Rightarrow S(v') \in E_\lambda(T)$ and thus 0

$E_\lambda(T)$ is invariant under S

b) Let λ' an eigenvalue of S, $E_{\lambda'}(T)$ associated eigenspace;

As $E_\lambda(T)$ is invariant under S, and $E_\lambda(T)$ has dimension 1, then every vector $v \in E_\lambda(T)$ can be described by $a \cdot v$, for $v \in F$. Thus

$S(v) = b \cdot v$, for some $b \in F$ ✓ 2

thus every $v \in V$ is in some $E_b(S)$ an eigenspace of S, thus

$E_\lambda(T) \leq E_b(S)$

c) If T diagonalizable and $\mathrm{gmult}(\lambda_i) = 1$ $\forall i$, then from the previous ques. we know 1

$E_{\lambda_i}(T) \subset E_{\lambda_i'}(S) \Rightarrow \mathrm{gmult}(\lambda_i) \leq \mathrm{gmult}(\lambda_i') \; \forall i$

Let $\{v_i\}$ a basis such that each $v_i \in E_{\lambda_i}(T)$, then this forms a basis of v_i, and

$$S(v_i) = \lambda_i' v_i$$

And thus, as each vector in $x \in V$ can be described as

$$x = \sum_{i=1}^{n} a_i v_i$$

✓ /2

Then

3. Let V be an inner product space over $F = \mathbb{R}$ where the inner product is denoted by $\langle x, y\rangle$ for $x, y \in V$. Consider the map

$$\ell : V \to V^*, x \mapsto (V \to F, y \mapsto \langle x, y\rangle).$$

(a) (2 points) Show that ℓ is linear.

(b) (2 points) Show that ℓ is injective.

(c) (2 points) Assume that V is finite dimensional. Let $\{v_1, \ldots, v_n\} \subset V$ be an orthonormal basis of V. Denote by $\{v_1^*, \ldots, v_n^*\} \subset V^*$ the dual basis. Show that $\ell(v_i) = v_i^*$ for all $1 \le i \le n$.

(d) (2 points) Describe how you could conversely use an isomorphism $T : V \to V^*$ to construct a function $V \times V \to F$ and under which condition this would be a inner product. (You do not need to prove anything here).

a) Let $x, x' \in V$, $a \in F$, $y \in V$, then

$$\ell(x+ax')(y) = \langle x+ax', y\rangle = \langle x, y\rangle + a\langle x', y\rangle = \ell(x)(y) + a\,\ell(x')(y)$$

and

$$\ell(0)(y) = \langle 0, y\rangle = 0$$

/2

b) Let $x, x' \in V$ s.t. $\ell(x) = \ell(x')$, then $\forall\, y \in V$

$$\ell(x)(y) = \langle x, y\rangle = \ell(x')(y) = \langle x', y\rangle \Rightarrow \langle x, y\rangle = \langle x', y\rangle$$

this is only true $\forall\, y \in V$ if $x = x'$ ✓ /2

c) let $y \in V$, then for ~~$a_1, \ldots, a_n \in F$~~

$$y = \sum_{i=1}^{n} \langle y, v_i\rangle v_i$$

thus, $\forall\ 1 \le j \le n$

$$\ell(v_j)(y) = \langle v_j, y\rangle = \sum_{i=1}^{n} \langle y, v_i\rangle \langle v_j, v_i\rangle = \sum_{i=1}^{n} \langle y, v_i\rangle \delta_{ij} = \langle y, v_j\rangle$$

and

$$v_j^*(y) = v_j^*\left(\sum_{i=1}^{n} \langle y, v_i\rangle v_i\right) = \sum_{i=1}^{n} \langle y, v_i\rangle v_j^*(v_i) = \sum_{i=1}^{n} \langle y, v_i\rangle \delta_{ij} = \langle y, v_j\rangle$$

for any $y \in V$, thus $\ell(v_j) = v_j^*$ $\forall\ 1 \le j \le n$

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